ANALYSIS 2000: CHALLENGES AND OPPORTUNITIES

L'analyse en l'an 2000: défis et opportunités

par Lynn Arthur STEEN

Dans cette contribution on essaie de tracer les lignes de développements possibles de l'enseignement de l'analyse. Il importe de souligner que dans le monde de l'éducation le mot "analyse" est plutôt utilisé au sens de l'américain *calculus*. Il s'agit de calcul différentiel et intégral conçu pour les masses; il se concentre sur des procédés et des recettes, alors que l'analyse s'intéresse aux preuves et aux définitions fondamentales.

En tout cas, l'analyse n'occupe plus à présent la position dominante qui était la sienne il y a cinquante ans. Quels sont les facteurs contemporains de nature à influencer le futur de l'enseignement de l'analyse?

D'abord on énumère des facteurs liés à l'environnement: facteurs sociaux (responsabilité publique, mutations technologiques, enseignement à distance, économie globale); facteurs liés à l'éducation (méthodes d'évaluation, nombre élevé d'étudiants, dont la préparation n'est pas homogène), facteurs mathématiques (diversification des mathématiques, "mathématiques pour tous").

A propos des réformes de l'enseignement du calcul différentiel et intégral, on mentionne deux mouvements qui se sont développés aux États-Unis. Il y a vingt ans l'intêret pour l'informatique a remis en question la suprématie de l'analyse comme porte d'entrée aux mathématiques universitaires. Ce mouvement a eu le soutien de scientifiques, d'administrateurs, de politiciens et de mathématiciens. Dans le même temps, mais indépendamment, le cours intitulé "*Calculus*" devenait un symbole de haute qualité de l'instruction: les parents comme les élèves revendiquaient pour ce cours un rôle central dans le programme du secondaire. Le but n'était nullement de *perfectionner* ce cours, mais de permettre au plus grand nombre de *passer l'examen*, si possible avec des notes élevées.

La réforme a changé les cours d'analyse plus dans la pédagogie et le contexte que dans les contenus. On met l'accent surtout sur les ordinateurs, les calculatrices de poche, le travail en groupes, les projets d'étudiants, l'écriture, la modélisation, l'établissement de liens et les applications. On demande aux étudiants d'avoir une plus

profonde compréhension des relations entre les formules, les graphes, les nombres et les descriptions verbales. Mais l'impact le plus important de la réforme de l'enseignement de l'analyse a été l'engagement des professeurs d'université dans la discussion sur l'apprentissage des mathématiques élémentaires destinées au plus nombre.

L'avenir de l'enseignement du calcul différentiel et intégral met en jeu plusieurs questions. D'abord il y a la manière dont les étudiants apprennent le sujet et les obstacles cognitifs. Par exemple, il faut évoquer le passage de l'algèbre à l'analyse, l'influence du contexte et des expériences quotidiennes sur l'apprentissage, le rapport entre l'éducation secondaire et tertiaire, le problème de la rigueur. Mais il est également important de se demander comment les mutations technologiques ont transformé l'approche de l'analyse. Comme les problèmes d'apprentissage sont difficiles à résoudre, peut-être les mathématiciens devraient-ils prendre connaissance des recherches qui se font dans les neurosciences sur les constructions mentales des mathématiques.

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by Lynn Arthur STEEN

Talking about challenges facing analysis education at the turn of the new century is a daunting task, well beyond the wisdom of any one person. However, it does provide a rare opportunity to think about what might be rather than what is, to imagine inquiries unencumbered by the constraints of current literature or prevailing orthodoxy. Thus my role is not that of a mathematician who proves theorems that last forever, but that of a prophet who imagines a future that is yet to emerge. I take some comfort in the knowledge that similar predictions on other anniversary occasions have set a very low standard for this genre.

Instead of beginning as a mathematician would, with definitions and notation, I shall begin, as an amateur prophet might, with clarifications and distinctions. Although my assigned subject is *analysis*, in the world of education analysis tends to mean *calculus*. Indeed, the vast majority of analysis enrollments are in elementary calculus. The distinctions between what mathematicians call analysis and what students call calculus are clear to anyone who has taught both courses. *Calculus focuses on procedures and templates, analysis on definitions and proofs*. Calculus is for the masses, analysis is for the 'mathes' — those who plan to specialize in mathematics. In this paper, my focus will be on calculus.

A second distinction is about mathematics vs. education. During the last half-century mathematics has expanded enormously, both in the diversity of its specialties and in the pervasiveness of their roles in society. These so-called *mathematical sciences* encompass a diverse and rapidly expanding part of human intellectual accomplishment. Although still vigorous both in its own right and as a supporting tool for other parts of mathematics, analysis now represents a much smaller fraction of mathematical practice than it did fifty years ago, before linear programming or digital computers, before combinatorics or bioinformatics, before data mining and string theory.

Paradoxically, during this same half-century the role played by calculus in education has expanded enormously, often irrationally. Today the mathematical focus of secondary school is calculus, not mathematics. Even as the mathematical sciences have built avenues of intellectual exchange extending in many different directions, the signposts of society still direct everyone to enter mathematics on the traditional highway of calculus. The question we must confront at the beginning of this new century is whether this traditional route still serves mathematics well or whether it might not be prudent to explore other options through which students can enter the world of mathematics.

1. Environmental scan

As is fashionable these days among those whom we in the United States suspiciously call 'policy wonks', I begin with an environmental scan to survey briefly some powerful contemporary factors that are likely to have significant bearing on the future of calculus. First are four societal factors:

- *Public accountability.* All over the world, government leaders are raising their expectation that universities serve broad public purposes, not merely elite professional interests. This demand arises both from increasing democratization of governments — who must now be at least somewhat responsive to the people they govern — and from rapid increases in secondary and tertiary enrollments that are supported to a great degree by public funds. Accountability pressures on mathematics departments are now external, not just internal; they arise as often from parents and politicians as from professionals and peers.
- *Technology.* As computers become standard tools of employment and research, proficiency in professional use of technology becomes an obligation of education. Mathematics educators now must deal not only with the question of how technology can enhance (or impede) mathematical learning and how technology changes priorities for mathematics content, but also how technology changes the way mathematics is expected to be performed. Technology's challenge to mathematics education is now about ends as well as means.
- *Telecommuting.* Mathematical skills, being purely cerebral, can be bought and sold anywhere on a worldwide market that is increasingly linked with high speed Internet connections. Thus the market for mathematical skills is no longer local or parochial but international. As graduates can now

sell technical skills to employers anywhere in the world, so students and teachers can also join a community of learners without borders. Thus despite lingering national differences in mathematics curricula, calculus as a subject is increasingly shaped by international contexts.

- *Global economy.* As nations compete in a technologically sophisticated international economy, the demand for technically trained employees who can manage complex industries is increasing far more rapidly than is the demand for researchers and scientists. Thus the economic demand for mathematical skills is now tilted in the direction of breadth rather than depth, for the practical skills of calculus rather than the theoretical skills of analysis, and even more for the data-based skills of statistics rather than the function-based skills of algebra.

Now three educational factors:

- *Diversification of education.* With the growth of a "parallel universe" [Adelman 2000] of higher education — on-line courses, for-profit colleges, certificate programs, and virtual universities — students now have many more options for their education. For better or worse, school and college mathematics departments will no longer have hegemony over calculus.
- *Assessment.* More and more, educational quality is being measured by outputs (student learning) rather than inputs (faculty teaching). Increasingly, calculus will be judged not by the syllabus, textbook, or instruction, but by the degree to which students can demonstrate that they meet the objectives of the course — many of which are totally absent from traditional tests.
- *Enrollment pressure.* In virtually every country the number of students completing secondary school and entering post-secondary education is rising, bringing to upper secondary and university levels students of very different skills, backgrounds, and motivations. These changes, worldwide in scope, increase enormously the pedagogical challenges of teaching calculus.

Finally, two mathematical factors:

- *Diversification of mathematics.* The expansion of mathematical methods into such diverse areas as genetics, finance, and even cinema has significantly changed the balance of mathematical practice. New tools developed for these new applications, especially in combinatorics, geometry, and data analysis, have displaced analysis from the leading role it has played for nearly three hundred years during which it has been at the center of the mathematical universe. As analysis no longer plays the lead role in mathematics, so calculus is gradually losing its claim to be the centerpiece of mathematics education.

- *Math for all.* Worldwide, educational and governmental leaders now include mathematics as part of the common core of learning expected of all students. However, especially in light of the changes in the mathematical sciences, one must now ask whether 'math for all' should continue to mean, as it has in recent decades, 'calculus for all'. There is some evidence that accelerated and excessively narrow mathematical requirements have created a backlash, leading some countries to scale back mathematical requirements in the schools.

2. CALCULUS REFORM

In the United States, and to varying degrees in other countries, one of the major factors influencing calculus in recent years has been a movement marching under the banner of 'calculus reform'. The progress and setbacks of this campaign provide a valuable case study in change, illustrating both how calculus responds to external pressures as well as how it manages to retain surprising equilibrium despite these pressures.

I apologize for focusing in this international forum on a movement that arose and remains anchored in the United States. I do this for both weak and strong reasons. The weak reason is that it is what I know best. The strong reason is that students and teachers in the United States come literally from all over the world. Thus the challenges we face are in some ways representative of the challenges faced by students and teachers in other countries. Indeed, our mixture of cultures in a single nation, a single city, a single school or university, even a single classroom, creates extraordinary instructional challenges that may be as internationally representative as anywhere on earth.

Twenty years ago, rising interest in computer science began to challenge the primacy of calculus as the gateway to university mathematics. Computing both pulled students away from calculus and challenged the importance of topics taught in calculus. One response was a campaign led by US college and university mathematicians to make calculus more "lean and lively", to make it "a pump not a filter" in the educational pipeline [Douglas 1986; Steen 1988]. The *reform movement* drew support from very different sources — from scientists who were frustrated by the inability of students to use calculus intelligently in real applications, from administrators who were angered by high failure or withdrawal rates from calculus courses, from politicians who saw in the global economy an increased need for technicians rather than theoreticians, and from mathematicians who recognized that calculus instruction had become, in Bernard Hodgson's memorable image, "*l'enseignement sclérosé*" [Hodgson 2000].

Simultaneously but independently, calculus in the United States also became a political totem, a supposedly objective and unassailable surrogate for high standards to which politicians could appeal in supporting or attacking various education proposals. Backed by political rhetoric that proclaimed calculus as the epitome of academic accomplishment, parents and students began pushing calculus into the secondary curriculum, primarily through the vehicle of the *Advanced Placement (AP) program*. This political campaign had nothing to do with reform but everything to do with status and access. Its goal was not to *improve* calculus but to enable more secondary school students to *pass* calculus, preferably with high grades (see, e.g., [Stewart 1997] and [Spence 2000]).

Needless to say, these two movements had rather different aims and objectives. The goal of the AP course soon became numbers : more offerings, more participants, more passing grades. The course itself is anchored by a traditional syllabus set by an external committee that also establishes the national exam for the course. Recently the syllabus was changed slightly to reflect some aspects of the calculus reform movement, but even these small changes produced great anxiety among AP teachers who depend on the course's stability and predictability for their success in getting students through the exam.

Even as AP calculus became the gold standard of secondary school mathematics, the tertiary level calculus reform movement tried with increasing energy to destabilize those entrenched aspects of postsecondary calculus that, in reformers' eyes, were not serving students well. For better or for worse, reform calculus became a magnet for unfulfilled goals of every prior progressive movement in mathematics education. Ten years after calculus reform had begun, a federally sponsored assessment of the movement found over a dozen different goals, only a few of which had anything to do with the content of calculus [Tucker & Leitzel 1995].

Three content goals, not among the first on the list, did imply fluency in traditional skills:

- Develop improved in-depth understanding of specific mathematical concepts.
- Employ calculus techniques in other disciplines and in novel situations.
- Reason analytically, qualitatively, and quantitatively.

Several cognitive goals stressed robust problem solving skills:

- Represent problems algorithmically, graphically, verbally, numerically, and symbolically.
- Use technology in solving problems.
- Deal with complex, often ill-defined problems.
- Translate problems from one form to another.

Some behavioral goals concerned how students work and interact with others:

- Communicate mathematical ideas both in writing and orally.
- Become independent mathematical learners.
- Work effectively in groups.

Finally, attitudinal goals focused on students' feelings about mathematics:

- Understand the role of experimentation, conjecture, verification, and abstraction.
- Develop positive attitudes about one's ability to do mathematics successfully.
- Appreciate mathematics' elegance and structure.
- Pursue further work in quantitative fields.

3. Evaluation and assessment

It is now fifteen years since the movement to reform calculus began, time enough to assess its short-term impact. It turns out that reform courses and textbooks — in the United States the text largely defines the course — do not differ much from traditional courses in content, but they do differ significantly in pedagogy and context. Reform calculus typically places much greater emphasis on calculators, computers, cooperative learning, students projects, writing, modeling, connections, and applications; it makes only modest changes in the way limits, derivatives, integrals, and series are approached or developed. Although evidence is scarce and unreliable, the consensus of experts and the conclusions of several studies is that the majority of students in the United States still study calculus from texts and in classrooms that have changed very little during the fifteen years of the calculus reform movement. However, a sizeable minority — typical estimates range from 25% to 40% — encounter calculus in courses that attempt to implement at least some of the strategies common to reform courses [Tucker & Leitzel 1995, 23–27; Roberts 1996, 157–8].

These students do face different expectations. As evidenced by final examinations, students in fully reformed courses, compared with their peers in traditional courses, are expected to have much deeper understanding of the relations among formulas, graphs, numbers, and verbal descriptions. For example, in a traditional US course, a typical exam problem at the end of the first term would give the formula for a function and ask the student to find the derivative and then sketch both the function and its derivative. In a reform course, a typical problem might give students six or eight unlabeled graphs and ask them to decide which are derivatives of which others and to explain how they figured it out [Roberts 1996, 74–132].

Convincing evidence of changes in students' mathematical performance, cognitive habits, work behavior, or attitude towards mathematics is both scarce and decidedly mixed. Since the goals of a reform course are so different from those of a traditional course, the lack of clear-cut evidence is not surprising [Schoenfeld 1997; Gold, Keith & Marion 1999, 229–256]. Different students learn different things well, and different things poorly; some thrive under the reform regimen, others chafe. Even good students often encounter difficulty when they shift from one style of course to another because it upsets their own expectations of what is required. Direct comparison of outcomes is difficult, if not nonsensical — a bit like comparing the outcomes of a course in Shakespeare with a course in James Joyce.

By far the greatest impact of the calculus reform movement is the engagement of university mathematicians, sometimes for the first time, in serious, productive discussions about teaching and learning elementary mass-marketed mathematics [Tucker & Leitzel 1995, 36–42]. Ideas for improved pedagogy spin off from these discussions and take root in other courses across the curriculum. Many university mathematicians have discovered that they share a common agenda with secondary school mathematics teachers who are working to implement new standards for school mathematics. Perhaps even more surprising, many university administrators find in the reform movement meritorious evidence of scholarship that both advances the art of teaching and resonates with institutional evaluation for tenure and promotion [Joint Policy Board 1994].

4. Inquiry and Scholarship

Calculus reform is one manifestation of efforts to improve the teaching and learning of analysis — and more broadly, of higher mathematics. Another is the growing interest in research in mathematics education at the upper secondary and undergraduate levels (where analysis education begins). These are not unrelated, of course, since most advocates of calculus reform claim to ground their recommendations in the results of research. (Not surprisingly, most critics of the reform movement (e.g., [Wu 1997; Askey 1999]) dispute the relevance of that same research.)

I tend to adopt a broad view of research, one aptly expressed by the phrase "disciplined inquiry" that Mogens Niss employed in his plenary address at ICME-9 [Niss, forthcoming]. This phrase nicely subsumes four distinct yet interconnected aspects of scholarship (of discovery, of synthesis, of application, and of teaching) that Ernest Boyer identified in his well-known monograph *Scholarship Reconsidered* [Boyer 1990]. The literature is filled with evidence of more focused or more idiosyncratic definitions, especially of educational research. I am not an expert in this area, but neither are the tens of thousands who teach calculus. So in this respect, I stand in their shoes as one who is more a practitioner than a theoretician of education. It is not natural for me, or for them, to ask about belief systems or metacognition, much less about APOS (action–process–object–schema) stages or statistically significant p-values (e.g., [Kaput & Dubinsky 1994; Dubinsky, Kaput & Schoenfeld 1994–1998]). Valuable as these may be to experts, they do not communicate in the language spoken by practitioners.

So instead of attempting to outline a research agenda cast in the traditional language of educational research, I want to suggest a variety of issues concerning the future of calculus that I believe could benefit in coming years from disciplined inquiry and reconsidered scholarship. Some of these issues are about cognition and understanding, some about teaching and learning, and some about policy and practice. Professionals may consider some of these issues out of the normal bounds of educational research. But I believe, and I hope you will too, that not only calculus but also analysis and, indeed, all of mathematics would nonetheless benefit from these kinds of disciplined inquiry.

Before delving into the issues, I need to make one more distinction — that between descriptive and normative questions, between inquiry into 'what is' vs. questions about 'what should be'. For example, in the history of calculus reform, one of the key results from the 'what is' inquiry was evidence of astonishingly high rates of dropout, failure, and repetition in calculus as taught in US colleges and universities. No other subject was even close. Another indicator, perhaps a bit more subjective, was the widespread sense among both mathematicians and users of mathematics that even students who completed calculus with good grades were, in too many cases, incapable of using it intelligently and expeditiously.

These failures of traditional calculus were documented by descriptive investigations. In contrast, reform calculus — the movement — was the result of disciplined inquiry into the normative question of what calculus *should* look like if we want to improve retention and improve performance in subsequent courses. As we have seen, this inquiry also led to additional goals, to cognitive, behavioral, and attitudinal expectations that were not central to the reform effort in its early, descriptive, hand-wringing stage. For some mathematics teachers, these newer goals — for example, writing, using technology, mathematical modeling, and group projects — are strategic objectives aimed at bringing about more traditional goals of improved retention and performance. For others these goals are ends in themselves, dimensions of what it means to learn calculus in the twenty-first century.

In most instances, descriptive and normative inquiries occur simultaneously on parallel paths. Ideas and evidence from one influence the other, generally for the betterment of both. Nonetheless, insofar as possible, I believe that it is important to keep the distinction clear, much as journalists, ideally, attempt to maintain a relatively clear line between reporting and advocacy. That said, I begin with some issues that I believe merit disciplined *descriptive* inquiry.

5. Descriptive issues

How many students study calculus? What do they learn? This is basic. In the United States, we gather some data addressing this issue, but it is relatively infrequent and not easy to interpret because much calculus in secondary schools is embedded in other courses, or taught superficially as a warm-up for later formal study [Loftsgaarden, Rung & Watkins 1997]. Other nations may do their own studies, but there seems to be no source of accurate international enrollment information on courses such as calculus that are optional and taught in many different kinds of educational settings. (And now we have the added complication of on-line Internet courses.) Calculus is one of relatively few optional courses of broad significance that is taught everywhere in the world, so it may merit special attention as a surrogate for the quality and extent of advanced education in different nations.

In what ways does calculus depend on its setting? This inquiry probes the universality of calculus by seeking explicit information about variability that may be due to external contexts: different nations, different educational settings (secondary school, technical college, university), different student abilities, different settings for diverse applications (physics, biology, business). What is common and what is variable? Are there any universals in terms of definitions, problems, theorems, applications, sequencing? Are there particulars that are either widely adopted or widely ignored depending on context? An international survey of calculus instruction could help define the subject by identifying what is central and what is peripheral.

How well do calculus tests assess the goals of calculus courses? This is an important empirical question because tests, not syllabi, determine what most students actually learn. In traditional courses in the United States, tests ask primarily for routine calculations plus one or two predictable applications and perhaps some simple proofs [Steen 1988, 177–211]. Exams for reform courses include a wider variety of conceptually interesting problems, often expecting fluency not just with formulas and functions, but also with graphs, numerical tables, and verbal descriptions [Roberts 1996, 74–132]. But in neither case does one find many questions that reveal the broader goals of reform calculus nor the deeper goals of traditional courses.

How important is calculus now that graphing and symbol manipulating software is widely available? Is calculus still as important as it was during the three hundred years between Isaac Newton and Bill Gates when it was the only tool available for most scientific models? Put another way, if digital computers had been invented before calculus, would calculus ever have gained the central position it now occupies in mathematics education? Now that virtually any set of differential equations can be solved digitally, closed-form analytical solutions are no longer the gold standard of mathematical modeling: visual representations of changes in behavior under different values of parameters offer far more insight than do symbolic solutions. This question calls for empirical investigation into the practice of mathematics to determine the degree to which the new digital empire has replaced the analysis empire that has been in power for most of the last three centuries. Results of this investigation will almost surely lead to a revised map of mathematics.

Which aspects of calculus require human expertise and which are best performed by computers? This inquiry into the content of calculus in the computer age naturally splits into two parts — calculus as it is practiced and calculus as it is taught. Long experience has provided mathematics teachers with a pretty good mental map of the logical and instructional sequencing appropriate for calculus in its traditional form when most student work is done with paper and pencil. But now that computer software can do much of what students have historically been asked to do, and now that computer systems are widely used in every profession where calculus plays a significant role, we need to envision calculus differently. Technology is doing for calculus what a new subway does for a city and what international air transportation is doing for the globe: it is realigning relationships and changing psychological distances. A new map of calculus would reflect these changed relationships.

What mathematical uses of calculators and computers are inappropriate? Twenty years' experience shows rather convincingly that calculators can, if wisely used, enhance students' experiences in mathematics class. In these masterful classes, students learn mathematics well, gain facility with a handheld mathematical machine, and emerge enjoying mathematics more than those who study the same material in more traditional contexts. However, university mathematicians universally complain that large numbers of students use calculators inappropriately — either to perform calculations that they really should do in their heads or on paper, or to give approximate numerical answers to problems where an exact answer (e.g. $\pi\sqrt{2}$) would reveal much greater understanding than the numerical approximation (4.4428829). Mathematicians are not alone in this concern; older adults often complain when they see young store clerks rely on calculators to perform simple calculations. So this inquiry is not about whether calculators enhance or diminish learning, but whether it is possible to distinguish between appropriate and inappropriate uses of calculators in a way that will be useful for teaching and learning. (If this could be done, then perhaps mathematicians might be able to reach consensus, now regrettably lacking, on the appropriate role of calculators in calculus.)

How does the way calculus is taught influence students' ability to use calculus in further study and professional practice? Most research into pedagogy is concerned about its relation to learning. This proposed inquiry points in a different direction by asking not about the effect of teaching practice on student learning but about its effect on students' subsequent ability to use what they have learned. Students' notorious inability to transfer learning from one context to another — from mathematics class to economics class, for example, or from classroom to work place — is widely recognized as a generic problem of learning, not a special disability of mathematics. Student resistance to employing good writing outside of language classes is equally well known. The issue for inquiry is whether and how the circumstances of learning can facilitate transfer of what has been learned.

How much of calculus is learned outside of calculus class? We know from many studies that young children pick up considerable mathematics outside of school: at home, in the playground, in stores, on television, and now at the computer. Children's mathematical minds are shaped substantially both by what happens in and outside of mathematics class. Historically, this has also been true for calculus students. Typically, students learned calculus deeply primarily by using it for physics. (And they learned algebra fluently primarily by using it for calculus.) Today relatively few mathematics students study physics, but many study business and economics where calculus-based models abound. We need to learn systematically (rather than anecdotally) just how important these out-of-math-class experiences are for full mastery of calculus. Does it matter much for mastery of derivatives and integrals if the primary application is to physics — the historic taproot of calculus — or to newer sciences such as economics or biology? More interestingly, can average students really learn calculus well if they study only calculus? Might it be that the subtle nature of infinite processes requires the anchor of a real model (rather than only mathematical definitions) for the mind to construct an appropriate and usable mental model of calculus?

What special cognitive hurdles are involved in the transition from algebra to analysis? All mathematics teachers recognize an enormous gap between the knowledge and skills developed in secondary school courses in geometry and algebra — knowledge that is primarily static and procedural — and the dynamic subtlety of limits and limit-based concepts such as the real number line, the derivative, and the integral. This gap is partly due to conflicts between ordinary and mathematical language (e.g., *limit* as a barrier; *continuous* as incessant; *infinite* as unfathomable), but even more to the increasing level of abstraction that is inherent in nested quantification (e.g., "for every epsilon there exists a delta such that ..."). It took mathematicians two centuries to figure out how to express the fundamental definitions of calculus in such a way as to avoid contradictory inferences when reasoning about infinite processes. It is not enough for calculus teachers to just understand the logical resolution of these paradoxes of the infinite — the so-called 'arithmetization' of analysis introduced in the nineteenth century. The issue requiring inquiry is how these logical resolutions may in fact exacerbate (rather than resolve) the psychological impediments students face in dealing with infinite processes.

What characteristics of students, teachers, schools, and policy account for the significant differences between secondary and tertiary education? Calculus, the introduction to analysis, straddles the active fault line where secondary education pushes up against the plate of tertiary education. Stress is evident everywhere — on students, teachers, schools — and discontinuities abound. Students face the daunting challenge of leaving the security of algebra for the uncertainties of analysis at the same time as they traverse the steep terrain leading from secondary to higher education. Although anticipation of educational and intellectual hazards will not eliminate them, foreknowledge can help both students and teachers develop strategies to minimize their negative effects. The purpose of this inquiry is to identify and describe special impediments to students' intellectual growth in these years due to factors other than the special character of analysis.

6. Normative issues

Shifting gears, I conclude with some normative questions, with suggestions for more speculative and subjective inquiry into possible futures for calculus, and *ipso facto*, for analysis education.

What is the best way to introduce students to university mathematics? In particular, is calculus the right course for the majority of students? No one disputes that calculus is both significant and sublime. But are these sufficient reasons to require that all students enter higher mathematics through this single gateway? Students in the social, behavioral, and life sciences — now the majority of clients of mathematics — need a much broader portfolio of mathematical methods, notably statistical, combinatorial, and computational. Other themes, such as optimization or modeling, that represent more widely applicable areas of mathematics can provide intellectual challenges equal to those of calculus as well as a synthesis of methods from algebra, analysis, geometry, and combinatorics. Might multiple gateways to mathematics help reverse the worldwide decline in students who specialize in university-level mathematics?

Should mathematicians welcome increasing numbers of students to calculus? Many of the complaints one hears from calculus teachers about students' lack of preparation or motivation are consequences of society's pressure, wisely or not, to push more and more students into higher levels of mathematics. Many mathematicians talk as if they would much prefer that mathematics had remained an elite subject for a select few rather than become a basic subject for mass education. Analysis (that is, calculus) is where this argument is joined, since everyone acknowledges that all students must study the other foci of school mathematics — arithmetic, algebra, and geometry. Might it be better for mathematics, and for students, if calculus were delayed until fewer (and more mature) students were enrolled ?

Is calculus the right course to introduce the rigor of analysis ? Calculus not only represents the crowning achievement of the age of science, but it also serves as a phase transition in students' mathematical passage from algebra to analysis. It is where the clear crystals of arithmetic, algebra, and geometry liquefy into the fluid ideas of limits, derivatives, and real numbers. Earlier, when calculus served a very limited population of motivated and screened students who were preparing for careers in mathematically based fields, it functioned with three main goals : to teach students the mechanics of calculus, to introduce some of its myriad applications, and to introduce epsilon-delta arguments. But as calculus has come to serve a much more diverse clientele, this latter goal has gradually disappeared : syllabi now focus on tools and applications, assessments stress procedures. Thus most students who study calculus experience only results and applications, not methodology or foundations. Determining the appropriate balance between the pragmatic and intuitive on the one hand and the formal and rigorous on the other hand is largely a matter of values, a reflection of the goals one wants to achieve in the course.

Should mathematics stake its future on calculus ? Mathematics continues to hold a place of privilege in an increasingly crowded curriculum. Unlike most subjects, mathematics is universally compulsory for school children and widely recommended long after it becomes optional. Increasingly, its special place is being challenged, especially by information technology. Historically, mathematicians (and others) could easily make a strong case for retaining a focus on calculus not just for instrumental reasons of convenience but also on intellectual, cultural, pragmatic, social, and scientific grounds. But now calculus faces two significant challenges — from technology that can best even experts on most calculus tasks, and from significant shifts in the balance of power among major fields of mathematics. Might there now be better exemplars to make the case for mathematics to an increasingly skeptical public ?

Should calculus look the same everywhere ? How much local context should be reflected in the content, applications, context, and pedagogy of a calculus course ? Can one expect the same textbook or syllabus to work as well in South Dakota as in South Africa ? This inquiry touches on one of the deepest epistemological issues of mathematics education — the degree to which mathematics should be taught as a pristine objective discipline independent of local culture and customs, or as one of many subjective aspects of culture that is tightly bound to problems of local interest. The former emphasis tends in the direction of pure mathematics, the latter in the direction of quantitative literacy. At risk of oversimplification, one might say that the former reflects an elitist perspective, the latter a populist agenda. Which is better for the future of mathematics?

How much technology should be taught in calculus class? As bank clerks use calculators (rather than traditional algebra) to compute interest and loan payments, so engineers use professional computer software (rather than traditional calculus) to solve analysis problems. These tools include symbol manipulating software, interactive visualization, mathematically active notebooks, applets, and modern web-based communication tools. In this age, calculus students can rightly expect not only to learn traditional procedures, concepts, and applications, but also to acquire modest fluency in using the professional tools by which calculus is now practiced. Notwithstanding these legitimate expectations, there remains a fundamental question of purpose and value that needs to be thoughtfully addressed: Should calculus in school prepare people technologically for calculus at work, or should it instead focus primarily on understanding fundamentals as preparation for further study? One cannot rightly assess the success of a course or program, much less design appropriate strategies, without first reaching agreement on its goals.

Should computer notation become part of standard calculus instruction? As silicon computers began taking on the task formerly carried out by human computers, programmers ran right into one of the major roadblocks that has historically created so much difficulty for mathematics students: ambiguous and two-dimensional notation. To make things work, programmers introduced new notation that could be typed on a standard keyboard and interpreted unambiguously by software that lacked human intelligence for guessing meaning from context. This new notation is ubiquitous: $*$ for multiplication, $\wedge$ for exponents, : for ranges of summation, $:=$ for assignment, etc. Yet mathematics texts continue, for the most part, to use only traditional notation that evolved to meet nineteenth century needs when computers were nowhere in sight. Should mathematics adjust its standards so that students learn a single unambiguous system that can be communicated readily by e-mail? Might the clarity of notation that helped computers also help students?

What is the appropriate balance of content and context? Discussions of mathematics curricula often take place along two relatively independent dimensions: more or less (abstract) mathematical content, and more or less (authentic) worldly context. Different approaches to calculus can readily be plotted on a content-context plane — some are high in one dimension and low in another, some are low in both (because they focus mostly on mechanics at the expense of both content and context). Few, perhaps none, are high in both content and context because that would require far more time and effort than students ordinarily have available. Choices must be made to achieve a suitable balance. Like so many other questions about calculus, the answer to this question reveals more about the values of the person who provides the answer than it does about the nature of the subject.

Should theories of cognition influence the way analysis is taught? For at least a decade, if not longer, those who actively pursue research in mathematics education have been exploring cognitive issues related to learning calculus. Much attention has been given, for example, to the 'APOS' stages in students' developing grasp of the concept of function: these are first seen as actions (calculating values), then as processes (reflecting on actions collectively), then as objects (encapsulated processes), and finally as schemata (classes of objects defined by shared properties). Functions have also been studied as representations of correspondences and of covariation among related variables, ideas that continually evolve in students' minds [Harel & Dubinsky 1992]. Not only functions but also other familiar mathematical objects — the number line, the meaning of equality, the idea of area, even the idea of number itself — undergo successive reconstruction as students pass through the phase transition between algebra and calculus [Tall 1991]. The issue I suggest for investigation is not which of these theories can be confirmed, but whether and to what degree any of these theories have significant impact on the effectiveness of teaching. Should it matter what theory of cognition, if any, a calculus teacher believes?

Should mathematicians pay attention to what neuroscientists are learning about the mental constructs of mathematics? Heretofore, all theories of mathematical cognition have been based exclusively on behavioral evidence — rather like theories of illness before the germ theory of disease. Now, however, physical and biological evidence from neuroscience is beginning to appear, evidence about electrical activity in the brain when it engages in mathematical thought [Butterworth 1999; Dehaene 1997; Lakoff & Núñez 2000]. Although this evidence is still primitive, it may help suggest or clarify possible explanations for what we observe about the struggles students have in learning mathematics. For example, it now seems clear that the well-known difficulty of mastering the multiplication table has physiological roots — since the words for numbers are handled in a different part of the brain than other words. Might we someday learn something about how the brain deals with limits or other abstractions of analysis? Might such insight improve the teaching and learning of calculus?

7. Summing up

The challenges facing analysis education at the beginning of the twenty-first century, and the second century of *L'Enseignement Mathématique*, are quite different from those one might have listed fifty or one hundred years ago. Calculus is now relatively more important in more students' lives even while it represents a relatively smaller fraction of the expanding mathematical universe. Calculus is no longer, as it once was, the unchallenged tool of applied mathematics; statistics, linear algebra, and combinatorial methods all have strong claims to the crown of mathematical utility. Moreover, calculus is not even the only option for accomplishing its etymological purpose, namely calculating with what used to be quaintly called 'infinitesimals'. Computers, thank you, can now do all those things perfectly well.

As we enter the new century, I suggest that we must be forthright in challenging all historic assumptions about the role of calculus in the mathematics curriculum. Take nothing for granted; put all options on the table. If calculus were not now entrenched, would we choose to give it such high priority in such a crucial place in the curriculum? Are there better choices to help attract students to the advanced study of mathematics? What benefits might ensue if secondary school mathematics were freed of the burden of preparing all students for calculus?

Answers we give to these questions may well determine the fate of higher mathematics education, whether it will be seen as the relic of a glorious past that has now been overtaken by new subjects such as information technology, or whether it will be seen, as it has been for most of the last century, as a conveyor of skills and understandings that are crucially important for all educated people. We dare not take the future for granted.

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