Studying the Relationship Between Students' Perception of the Mean and Their Understanding of Variance

Robert Sigley[1], Layla Guyot[1], Alex White[1]

[1]Texas State University, Department of Mathematics

Abstract
This paper explores how Introduction to Statistics students think about and compare the mean and variability of four datasets. They explored the datasets through various representations (e.g., balance beam, leveling off) and ranked the datasets from most to least variance. When exploring the mean, the students found value in both approaches, but preferred the balance beam approach for variability. However, when reasoning about the balance beam they focused on the wrong properties and make faulty inferences. When reasoning using the leveling off representation, they focused on the correct properties and used them to make sound inferences about the data.

Key Words: Statistics Education, Content Knowledge, Mean, Variance, Representations

1. Introduction

A key idea in an Introduction to Statistics course is the mean as not only a measure of central tendency but as a measure of variability. Policy documents (e.g., The Gaise Report) stress the importance of students having multiple conceptions of the mean such as a measure of center and as a balance point. Research has found that representations such as a balance beam are useful in helping students understand the mean (Hardiman et al., 1984). Commenting about variation, Pfannkuch and Reading (2006) said, "variation is at the heart of statistical thinking…reasoning about variation is enabled through diagrams or displays" (p. 4). In this paper, we intend to explore how a student's conception of the mean influences their thinking about variation by having students view the mean through some of the popular representations (or displays) and seeing what features of the representations are they attending to when trying to determine the mean and variance of a dataset.

2. Literature Review

2.1 Conceptualizations of the Mean

Several conceptualizations of the mean appear in the literature, such as: (1) the fulcrum, (2) algebraic form of the Least Squares, (3) geometric forms of the Least Squares, (4) as a vector (Watier, Lamontagne, & Chariter, 2011). Two popular representations used for those concepts are the balance beam and leveling off. In the balance beam representation, the data points are laid out on a number line and the goal is to place a fulcrum on the number line, so it "balances" and does not tip in a certain direction. Students can explore what happens as they move the fulcrum and as they add more data points to the number line. This is a popular approach in K-12 classrooms and groups such as NCTM recommend its use. In the leveling off representation each number corresponds to a bar related to the length and the mean is located where the bars "level off" (i.e., there is as much above the bar as missing below it). This concept can help students realize that the sum of the residuals is zero. The leveling off approach is presented in popular textbooks for elementary pre-service teachers (e.g., Beckmann, 2017).

2.2 Research on Variance

The literature on variance has focused on topics such as reasoning about different distributions and comparing different data sets (Shaughnessy, 2007). Research with elementary aged children has shown that when interpreting and comparing datasets values such as the mode attract their attention and then they reason about the variance using the mode as a reference point (e.g., Konold, Higgins, Russel, & Khalil, 2015). Letting students look at dynamic views of the data using technology aids in helping students develop concepts of variation (Lehrer, Kim, & Schauble, 2007). While there is a body of research on both the mean and variance, they tend to be studied as standalone topics even though a robust understanding of the mean should help one understand variability. We explored the relationship between mean understanding and variability by having students work on a task that involves comparing variability using different representations of the mean. Our research question guiding our analysis is: What do students attend to when looking at a balance beam representation and a leveling off representation of the mean and variance of a dataset?

3. Methodology

This study took place at a large four-year college in the southern part of the United States. The participants (n=7) are students who were taking an Introduction to Statistics course during the time of the interview, which took place at the end of the semester after the course ended. They engaged in an hour long video-taped task-based interview (Maher & Sigley, 2014) with two of the authors where they: (1) describe what they thought the mean and variance are, (2) identify the mean and variance of a series of histograms (Figure 1) and then ordered the histograms in terms of least variance to most, (3) engage in a task that had them construct a distribution on a line using Unifix cubes and then move the cubes to show a distribution that would have more and less variance than the one they constructed, (4) use a program developed in Mathematica (White, Straughn, & Guyot, 2016) to dynamically explore different interpretations of the mean (i.e., balance beam, leveling off, the sum of squares, and as a projection), (5) re-rank the initial histograms (from least variance to most) based on the different interpretations (Figure 2), and (6) select one interpretation they had the most trust in being correct. While the participants were engaging in the task, the interviewers probed their thinking and asked them to justify their reasoning.

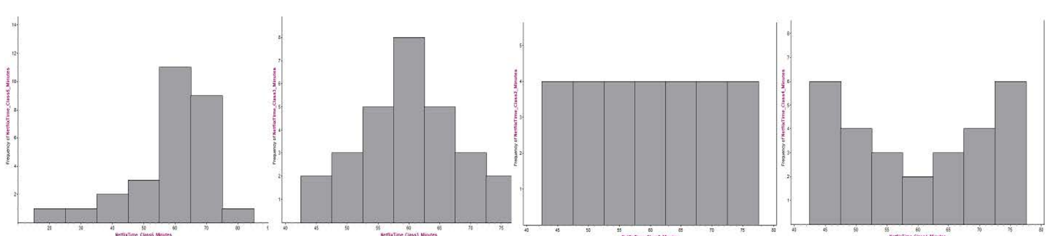

Figure 1: Histograms of the four distributions (skewed, normal, uniform, & inverted bell curve).

The authors watched the videotaped interviews and coded for critical events (Powell, Francisco, & Maher, 2003) around their justifications for why they ordered the graphs the way they did and how their interpretations of the mean related to their understanding of variance. In this paper, we are reporting on the participants responses and rankings around the final task (Figure 2) with a focus on their interpretation of the balance beam and leveling off representation. We will first present the ranking data for all the participants and then talk about how one student, Michelle, described her thinking about the two representations.

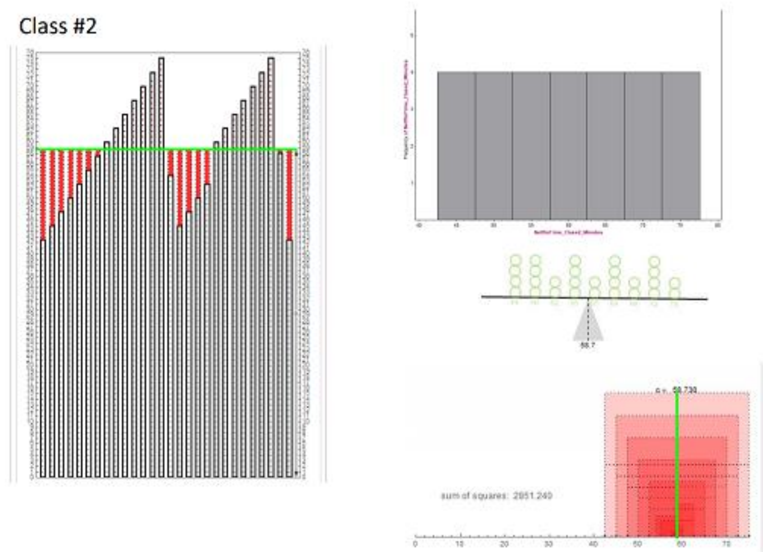

Figure 2: Second ranking task with the different interpretations from the Mathematica tool (leveling off, histogram, balance beam, and the sum of squares).

4. Results

4.1 All participants

We are presenting the student rankings for the leveling off and balance beam as those were the representations the participants stated they preferred and both are common approaches described in the literature. In the tables, the datasets are ranked on a scale from 1 to 4, with 1 containing the most variance and 4 the least. This was the scale the students used when engaging in the task. When evaluating the graphs, the students were given the opportunity to start with any of the representations they wanted and to go about choosing the others in any order.

Table 1 displays the rankings of the seven participants for the variance of each dataset only focusing on the balance beam representation of the data. Four of the seven students ranked the skewed graph as having the least variance, when it had the most. None of the students thought the normal distribution had the least variance and there was disagreement across the students on the ranking of the inverted bell curve dataset. Not one student correctly ordered the datasets based on the balance beam.

The data in Table 2 is the participants rankings for the datasets based on only looking at the leveling off representation. Based on that representation, every student identified the skewed dataset as having the most variance. Three of the students thought the normal dataset had the most variance, two correctly identified the uniform dataset, and four for the inverted bell curve. Two of the students ranking matched all four datasets.

Table 1: Student rankings of variance based on the balance beam representation.

Student	Uniform	Normal	Inverted	Skewed
Adam	3	2	1	4
Karen	4	2	3	1
Michelle	1	3	2	4
Chris	2	3	1	4
Teddy	2	3	1	4
Siobhan	4	2	3	1
Scarlett	4	1	3	2
Actual	3	4	2	1

Table 2: Student rankings of variance based on the leveling off representation.

Student	Uniform	Normal	Inverted	Skewed
Adam	4	3	2	1
Karen	4	2	3	1
Michelle	3	4	2	1
Chris	4	3	2	1
Teddy	3	4	2	1
Siobhan	4	2	3	1
Scarlett	2	4	3	1
Actual	3	4	2	1

4.2 Michelle

At the start, the interviewer asked Michelle which representation would help her understand the mean and variance better. She preferred the balance beam to help her understand both. For the mean, she preferred the balance beam as it was "easy to visualize the information [it] is right there" and for variance "it tells you the minimum number and maximum number exactly". In explaining why the skewed graph had the least variance she said "I tried to see which one had the variables more spread out, like fairly, this one had it [pointing to the mode of the skewed graph] and these have less [pointing to data points that have one or two balls on the beam]... [it has] like symmetry, we have one here [pointing to a data point to the left of the mode], and we have one over here [pointing to a data point to the right of the mode which is about the same distance away as the previous point]". For her the normal distribution had the second least variance because there is a lot of points in the middle and the other data points are symmetric around that. She claimed the uniform distribution, while symmetric, had the most variance because there "was not one number that had a lot of balls [data points]". Three other participants (Adam, Chris, and Teddy) shared Michelle's thinking where they liked the balance beam as it let them see the numbers appeared the most and then they could focus on the symmetry around that.

When attending to the leveling off representation and the mean, she said "once I understood what I was looking at and why some numbers went above the line [mean] and some below, the mean made sense to me", but she still preferred the balance beam approach as it was "hard to see how many times a number came up". About the variance, she claimed she was also looking at the "range" but she indicated that she was talking about the distance from the mean saying "this one [the skewed graph] has the most variance because all of these numbers are, like, so far from this line, I mean the mean, and some are not that far from the mean, but like, there are a lot that are really far from the mean and this one [pointing to the normal graph] has the least variance because, like, a lot of the points are not that far from the mean and only two [pointing to the two farthest points from the mean] are really far away and it's not even that far compared to these [pointing to several points on the skewed leveling off representation]".

After ranking all of the representations from least to greatest variance across the multiple representations, the interviewer pointed out to Michelle that she was inconsistent across her rankings for the balance beam and the leveling off and probed her to explain why. She said with the leveling off "I pay attention more to the picture" while with the balance beam approach "I pay more attention to the number". When asked which one she would rely on more, the pictures or the numbers, she said "the numbers - you always trust the numbers... I would trust the numbers more than the pictures" though when explaining why she would trust the balance beam more she referred multiple times to the shape of the data on the balance beam.

5. Discussion

When determining the mean of a dataset, the students preferred the balance beam approach, but saw value in the leveling off approach and said it was also useful in finding the mean. They reported their preference for the balance beam due to being able to "see the numbers". However, when interpreting the datasets variability using the balance beam representation they focused on the mode of the data and the symmetry of the data around the mode, but did not consider the mean of the data. This led to issues in ranking the datasets such as the skewed one and is like the issues the students display in the Konold et al. (2015) study. When considering variation through the leveling off representation, the students focused more on the distance of the data points from the mean, which led to them making correct inferences about datasets variation. While both representations were valuable in understanding the mean, the leveling off approach has advantages in helping students use that knowledge to understand variability.

Acknowledgements

Thank you to the Texas State University Research Enhancement Program for providing funding for the study.

References

Beckmann, S. (2017). *Mathematics for elementary teachers with activities* (7 ed.): Pearson Education.

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) report*. Retrieved from Alexandria, VA

Hardiman, P. T., Well, A. D., & Pollatsek, A. (1984). Usefulness of a balance model in understanding the mean. *Journal of Educational Psychology*, 76(5), 792.

Konold, C., Higgins, T., Russell, S. J., & Khalil, K. (2015). Data seen through different lenses. *Educational Studies in Mathematics*, 88(3), 305-325.

Lehrer, R., Kim, M. J., & Schauble, L. (2007). Supporting the development of conceptions of statistics by engaging students in measuring and modeling variability. *International Journal of Computers for Mathematical Learning*, 12(3), 195-216.

Maher, C. A., & Sigley, R. (2014). Task-based interviews in mathematics education. In *Encyclopedia of Mathematics Education* (pp. 579-582). Springer, Dordrecht.

Pfannkuch, M., & Reading, C. (2006). Reasoning about distribution: A complex process. *Statistics Education Research Journal*, 5(2), 4-9.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, 22(4), 405-435.

Shaughnessy, J. M. (2007). Research on statistics' reasoning and learning. In D. Grouws (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 957-1009). Reston, VA: National Council of Teachers of Mathematics.

Watier, N. N., Lamontagne, C., & Chartier, S. (2011). What does the mean mean?. J*ournal of statistics education*, 19(2).

White, A., Straughn, M., & Guyot, L. (2016). Different Perspectives on Achieving the Mean of a Distribution. Wolfram demonstations project. Retrieved from http://demonstrations.wolfram.com/DifferentPerspectivesOnAchievingTheMeanOfADistribution/