Mathematics for a New Century

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Mathematics is an ancient discipline vested with modern authority. Mathematics empowers people with the capacity for control in their lives; it offers science a firm foundation for effective theories; and it promises society a vigorous economy. In all cultures, in all generations, children study mathematics to gain access to a better life. In this respect, today is no different than yesterday, the twentieth century no different than the eighteenth.

But in other respects the world has changed—significantly, profoundly, and irreversibly. Until recently, school mathematics (mostly arithmetic) sufficed as the language of commerce, while higher mathematics was viewed as the language of science. No more. Today international commerce is based on computer models of trading that employ such sophisticated tools as stochastic differential equations, while medical research relies nearly as much on mathematical models as on clinical evidence. Indeed, the language of advanced mathematics—from control theory to combinatorics, from differential geometry to statistics—has permeated business, medicine, and virtually every information system of modern society.

Computer-based communication has created a world economy based as much on exchange of data and ideas as on shipment of goods. In industrialized countries, as much as half of the labor force is now engaged in information-based work, while in developing countries information-based industries are often the fastest growing segment of the economy. Mathematics is part of this new world order, an amplifier of the mind that bestows significant personal and economic advantage on those who possess it.

Implications for Mathematics Education

The new reality in which we all live, this high-tech prelude to the next century, augurs important changes for the context in which mathematics is taught in universities around the world. Different countries will experience these changes at different times and in different degrees. But none will escape the winds of change unleashed by the power of modern computers and innovative applications.

One might naively expect that as mathematics increases in its applications, good students would flock to mathematics in increasing numbers. But what we observe is quite the opposite: high demand mixed with low quality threatens post-secondary mathematics

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in almost every country. Before trying to understand this paradox, let's examine the evidence:

1. Shortage of Secondary School Teachers.

As demand for mathematics rises, demand for instruction in mathematics also rises. But this same demand attracts graduates to jobs other than teaching, because pay and job conditions are often much better outside educational institutions. The combination of increased enrollments with decreased numbers of prospective teachers creates shortages which are already being felt in many countries:

- *From Western Europe:* "Fairly soon, secondary school mathematics will be taught by people who are absolutely not qualified. It isn't yet as bad as the United States is right now, but it's pretty close."
- *From a Developing Country:* "Many mathematics teachers in our secondary schools have not taken *any* post-secondary studies."

2. Weak Mathematics Preparation of Entering University Students.

Two factors account for this trend, which is widespread in all regions of the world. Wherever secondary school mathematics teachers are poorly prepared, their students are likely to be poorly prepared. Even in countries with well qualified mathematics teachers in secondary schools, the many attractive new career options in computer science, business, and medicine often distract good students away from serious mathematics even before they enter the university. When careers in mathematics become only a second or third choice for many able students, university mathematics departments are bound to notice both a decline in the average preparation of entering students and, perhaps more important, an increase in the variance:

- *From Western Europe:* "The wide range of qualification levels at the end of secondary education prevents a uniform starting level in post-secondary education."
- *From Eastern Europe:* "The growing difference in preparation presents some problems at the university."
- *From a Developing Country:* "Since about 1970 the general level of mathematics preparation has been decreasing, and right now is *very* inadequate."

3. Declining Student Interest in Mathematics.

The very professions that use mathematics extensively provide attractive and financially desirable alternatives for expression of mathematical talent. For most students who want to make a contribution to society, traditional pure mathematics appears to be of limited interest when competing with more visible careers such as engineering, medicine, computer science, and finance. Careers in these more "practical" fields now pose stimulating challenges to the mathematical interests of the young student, comparable in intellectual reward and attainment to traditional mathematics.

- *From Western Europe:* "Our most serious problem is the attraction of engineering and computer science at the expense of mathematics."
- *From a Developing Country:* "Intelligent students now choose areas such as health sciences and engineering; mostly second or third rate students choose to study mathematics."

The evidence is clear from around the world: As mathematics becomes more applicable, able students flock to the disciplines that use mathematics rather than to traditional pure mathematics. Student preferences pose a direct threat to the health of post-secondary education—where student choices are most noticeable—and destabilize secondary education as well by reducing the number of well-prepared mathematics teachers. In extreme situations, declining student interest in mathematics can lead to chronic shortages of well-prepared teachers, thereby establishing a negative feedback loop which rapidly drives the whole system of mathematics education into a state of crisis. It is as if we wired our home thermostats to shut off whenever it got too cold: as fall turns to winter, we'd soon have no heat whatsoever.

Standards of Stability

Comparisons with the past are of limited value when analyzing problems of today. It is too easy to fall prey to the merchants of nostalgia who wish for restoration of a dream from the past. What's needed instead are standards by which the post-secondary system can be judged, independent of particular transient effects. Data from many countries around the world reveal some patterns that appear to transcend the great differences in educational systems. For example:

- In most industrialized countries, between two and three out of every million citizens earn a doctor's degree in mathematics. Only in developing countries and in the United States are doctoral productivity patterns significantly below this level.
- About 3% of university degrees are in mathematics, in both industrialized and developing countries. Despite wide variation in the availability of university education, in virtually all countries this percentage varies only from 1.5% to 4%.
- Most countries produce annually from 20 to 40 university mathematics graduates per million in the population, although a few industrialized countries are much more productive, with averages in the range of 60 to 80.

For a hypothetical country with a stable population, universal secondary education, and university education for 20% of the population, one would expect for each million citizens to need about 25-30 university graduates each year just to maintain the supply of secondary school teachers, and about 3 doctoral graduates to replenish the supply of university faculty.

Although neither these data nor estimates are based on very firm evidence, they do suggest that the norm in most industrialized countries at this time is that the educational systems produce just about enough mathematics graduates to replace themselves, with little left over to support an increasingly mathematicized society. It is not surprising, therefore, that in the face of increased demand for mathematically trained graduates, most nations are reporting shortages of teachers and weaknesses among mathematics students.

Changes in the nature of mathematics and society—increased usefulness of mathematics, increased mathematization of society—have shifted significantly the delicate balance between supply and demand of mathematically-educated university graduates. This shift suggests the need for structural changes in the way mathematics is taught in institutions of higher education, to insure that post-secondary mathematics programs continue to meet the increasingly diverse needs of society. The direction of change will be determined by the impact of several significant forces that influence university mathematics programs: computers, new applications, research in learning, research in mathematics, and socio-economic trends.

Impact of Computers

The most obvious force for major change in university mathematics is the rapidly increasing impact of electronic computers on the way mathematics is practiced. Computers have influenced mathematics in two quite different ways: they have made mathematics more powerful than ever before, and they have altered the very nature of the mathematical sciences. Computers are now commonly used in mathematical research of all kinds, both pure and applied; they have altered the balance among subdisciplines, posed new problems for theory, and provided new tools for exploration and proof.

Unfortunately, this new intellectual balance within the mathematical sciences is rarely reflected in school or university curricula. In most countries, computers have had only a very slight effect on post-secondary mathematics education, and then only in a few industrialized countries. Indeed, many countries report *no* significant post-secondary curricular change in the last 25 years—a period during which the entire computer revolution has occurred. University mathematics curricula are virtually immune to change; in mathematical terms, we might say that the mathematics curriculum is invariant under intellectual revolutions.

It is surprising that computers have had so little impact on the university mathematics curriculum. Lack of hardware, a very real problem in many countries and many universities, is more a symptom than a cause. The real issue lies much deeper: a lack of will.

Mathematicians are notoriously conservative in curricular matters, and resist changing their courses even if they regularly use newer techniques in their own professional lives.

Apart from actual use of computers, the impact of algorithmic approaches to mathematics cries out for a prominent role in the undergraduate curriculum. Such subjects as computational complexity, dynamical systems, scientific computation, and visual data analysis have blossomed in the last quarter century, but are hardly noticeable in the current curriculum. Much could be done to introduce these topics early even without actual use of computers. But most faculties don't make that effort, preferring instead to rely on the comfortable classical curriculum. Is it any wonder that good students no longer find mathematics an attractive subject?

Impact of New Applications

A second factor that suggests the need for major revitalization in undergraduate mathematics is the dramatic increase in breadth of application of the mathematical sciences. No longer are engineering and physics the prime users of undergraduate mathematics. Sophisticated techniques from yesterday's research are routinely used in today's applications.

Consider the biological sciences: differential equations are used in physiology, as are combinatorial methods in genetic sequencing, knot theory in modelling DNA, graph theory in neurophysiology, mathematical modelling in protein engineering, statistical methods in clinical trials, and probability theory in epidemiology. The list of new applications could go on indefinitely: mathematical biology is one of the most exciting frontiers of applied mathematics today, a great showcase for the power and versatility of mathematics. These mathematical tools have helped push biological research to the frontier of science: understanding life and intelligence is *the* scientific challenge of our age, and mathematics plays a central role in this venture just as it did a century ago in the quest to understand the nature of matter.

Similar mathematical methods are used increasingly in environmental science, in natural resource modelling, in economics and sociology, in psychology, and in cognitive science. In these newer disciplines, mathematics offers structures for understanding (and hence controlling) many of the factors that affect the quality of life on earth. Mathematical ideas are even finding their way into the fine arts, as computer tools are used by graphic artists, film makers, and musicians.

New applications have unquestionably changed the character of the modern mathematical sciences. They have changed not only how mathematics is used, but also the problems on which mathematicians work. Science has always inspired much of the most interesting mathematics, and this decade is no exception. Especially because of the natural attraction of diverse applications to so many students, it is perhaps surprising that these applications have had so little impact on the post-secondary curriculum.

The answer, once again, seems to lie in the natural curricular conservatism of university mathematicians. By maintaining a curriculum from the 1950's that is narrowly focused on traditional topics from mathematics with applications to engineering and physics, we give students the impression that mathematics is a discipline rooted more in the past than in the future. Nothing could be further from the truth, although one cannot blame students for not knowing this. We never tell them.

Impact of Research on Learning

Those who study the way students learn have accumulated convincing evidence for something that observant teachers have known for generations—that students retain more of things they learn by themselves than whatever they memorize for a test. Effective teaching requires that students be engaged by what they learn, not only that they pass tests based on effort of preparation.

The conclusion of this research should not surprise any experienced teacher. What's new, however, is evidence suggesting a mechanism that helps explain why learning takes place this way. Teachers normally act as if each student's mind is a blank slate—or an empty computer disk—on which effective teachers can record whatever information they like. Research in cognitive science suggests otherwise: each student's mind is more like a computer program than a computer disk. Each student brings to the mathematics classroom a rich set of prior mathematical experiences that provide a unique mental framework in which the student creates new patterns derived from new experiences. Learning occurs not in the act of remembering, but in the gradual development of mental frameworks unique to each individual. In other words, students learn by modifying their mind's program, not by storing new data in their mind's memory.

Some of each student's experience is based on school lessons, but much of it is learned from the ambient environment in which the child lives. Those at this Congress who are discussing ethno-mathematics can document example after example of student-generated mathematical algorithms derived from street experience rather than from school. Scientists who work with students in laboratory work know how much more is learned with the hands than with the mind alone. Students who work with computers learn incredible detail that they have never been taught simply by interaction with the machine. Learning requires engagement, not just activity.

I hardly need point out that most post-secondary mathematics—indeed, most mathematics at any level—is taught by lecture, with homework exercises for practice and examinations for enforcement. Lecturing and examining may be the easiest way to teach mathematics (or perhaps just the cheapest way), but they are by no means the most effective. Few students can learn mathematics well from lectures and homework alone. In the past, those few may have been all the students in our classes, sufficient in number to meet the needs of society. Today these few are the teachers and professors.

But now, with increased need for mathematics by students of many different interests, the deficiencies of this style of pedagogy are increasingly apparent. Effective teaching for today's students requires a more diverse repertoire of approaches, including in addition to lectures, homework, and examinations, new opportunities for group work, for extensive writing, for oral practice, for exploration and experimenting, for modelling projects, and for computer activities. Students need to experience mathematics as they learn it, and not simply study it in preparation for exams.

Impact of Research in Mathematics

Despite its ancient roots, mathematics continues to grow and change. What we call classical mathematics—in theory and applications—is largely the mathematics of the nineteenth century. In the first two thirds of the twentieth century the emphasis of mathematical research shifted to abstraction and axiomatic foundations, even as applications of mathematics to the physical sciences reached the point of what Eugene Wigner called "unreasonable effectiveness." Virtually all mathematics taught in university courses today is this classical mathematics of the nineteenth and early twentieth centuries.

However, in the last thirty years, mathematics itself has undergone a renaissance as creative and breath-taking as the parallel but better known revolutions in biology and computer science. These discoveries have affected both theory and applications, changing significantly the very character of mathematics:

- *Number Theory.* The digital character of computation has thrust number theory once again into center stage of mathematics, with new approaches to classical problems (for example, elliptic curves) producing unexpected dividends in the theory of computation, in mathematical logic, and in applications to the security of data transmission.
- *Statistical Science.* As data deluges science from every source— from economics, from telemetry, from laboratories—statistical methods form a continually changing interface between information from the world and analysis by mathematical methods.
- *Optimization.* Beginning with the simplex algorithm for linear programming and moving through recent interior methods devised by Karmarkar, mathematical optimization has joined the frontiers of mathematics both in challenging problems (to find optimal solutions) and in effective applications (to improve efficiency and reduce waste).
- *Visualization.* Converging ideas from research in perception, in computer graphics, and in geometry are now producing innovative theories of perception that draw on such diverse specialities as differential geometry, combinatorial algorithms, data structures, and engineering.
- *Dynamical Systems.* Recent research on fractals has shown how iterated order often begets chaos. Models of chaotic systems have permeated science and spawned whole new areas of research in mathematics; applications range from data compression in storage of photographs to theoretical models of instability (in stock markets as well as in the solar system).
- *Decision Theory.* Whereas continuous models from analysis provide apt models for the physical sciences, the discrete models of game theory, social choice functions, and expert systems provide more appropriate tools for the human sciences which depend on decisions, votes, and choices rather than on continuous change.

During the last thirty years, mathematics has been transformed into a rich collection of mathematical sciences, tightly coupled to each other through interlocked theory and linked to the world of science and business by an increasing network of applications. Unfortunately, few university students see mathematics in this way—as a vibrant, living discipline ready to engage their minds and careers in a stimulating contemporary challenge. More often than not, students see mathematics as part of history, as a collection of tools developed in some previous era whose scientific problems are now passé.

Impact of Socio-Economic Factors

It is no longer true in any country, if it ever was, that universities can function as islands of intellectual innocence amid an ocean of turbulent social change. All over the world, governments and students are increasing the pressure on universities to better meet the needs of society. Often these pressure are conflicting, overwhelming, and contrary to the traditional mission of universities. Like all subjects, mathematics is weakened when the strains on the universities become too great.

Four issues seem widespread in reports from around the world:

- *Overcrowding.* Public pressure for increased enrollment is common in virtually every country. Sometimes this pressure is caused simply by increases in the number of university age students; other times it is part of a conscious government decision to increase access to higher education as a means of furthering the educational expertise of the country. Moreover, since jobs are often hard to find while the status of "student" is frequently a comfortable refuge from adult responsibility, students often pursue very leisurely routes to their degrees. It is not uncommon in some countries for the majority of students to never finish.
- *Underfinancing.* Some countries are struggling with run-away inflation; others are insisting on stricter accountability from higher education for producing graduates useful to society. In either case, universities everywhere are under considerable financial pressure. Severe cutbacks are not at all uncommon. Because of the increased world-wide demand for mathematics, these problems often divert mathematically trained personnel from one country to another, sometimes causing "brain-drain" that reduces the intellectual resources of those countries that can least afford the loss.
- *Links to Jobs.* Many countries now experience the paradox of unemployed university graduates at the same time as positions requiring university degrees remain unfilled. Most often this is due to the difficulty of properly matching university work with job requirements; neither centrally planned economies nor free market systems seem able to solve this problem efficiently. Problems arising from socio-economic circumstances are significantly aggravated by a university mathematics curricula that typically trails by fifty years the mathematical methods actually used in contemporary business and science.
- *Diminished Quality.* As numbers prevail over standards, universities everywhere find that students are either unable or unwilling to undertake the type of rigorous curricula that have traditionally been the hallmark of university education. Elitism clashes with populism more directly on a university campus than anywhere else in society. Higher education is the crucible for this debate, and mathematics, to the extent that it provides the key to opportunity, is the principal fuel that heats this crucible.

Revitalizing Post-Secondary Mathematics

Signs from around the world make clear the urgent need for revitalization of post-secondary mathematics education:

- To attract good students to mathematics;
- To encourage potential mathematics teachers for secondary schools;
- To provide mathematical expertise for industry and government;
- To support advances in science and engineering;
- To help improve secondary education in mathematics.

Many strong forces influence university mathematics education. We have touched on only a few: computers, applications, research in learning, research in mathematics, and socio-economic factors. Because these forces are so strong, change in post-secondary mathematics can only be accomplished by channeling these forces to constructive ends. Yesterday Geoffrey Howson advised us to build a surfboard—instead of continuing vain attempts to stem a tide generated by uncontrollable forces. I prefer a slightly different metaphor: a sailboat tacking against the wind. Like a captain of a sailboat whose course is set against the prevailing wind, mathematics educators must learn to use the winds of change to our own advantage.

The most important task for post-secondary mathematics is to make it attractive as a subject of study for students with a wide variety of interests. University mathematics is good preparation for many different careers. Students these days see and expect personal and scientific challenges in many other fields, so they expect it also of mathematics. Those of us who teach in universities must:

- Teach modern mathematics as well as classical topics, not just to advanced students but also to beginners;
- Display the power of mathematics not only through its logic but also through the enormous variety of its applications;
- Show students that computers are a natural ally in mathematical practice—as a source of ideas, a device for visualization, and a tool for calculation;
- Engage students in the mathematics they study by developing new instructional strategies that involve students as active apprentices in the craft of mathematics;
- Make mathematics a pump rather than a filter in the educational pipeline, to insure that students of many different interests benefit from university study of mathematics.

Such changes will have significant beneficial effects on university mathematics programs, and by extension on school mathematics, as well as on science and society. Curriculum revitalization must involve new approaches, new priorities, and new examples that reflect not only mathematics of past centuries, which we all studied, but also mathematics for a new century, which our students will need.