Mathematics, Numeracy, and Statistics

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In my first talk I outlined general issues surrounding quantitative literacy—what it is, why it is important, and how it may be helpful in addressing some of the pressing problems in mathematics education. In this talk I would like to look more specifically at how quantitative literacy fits into the secondary school curriculum—specifically, at how it compares with mathematics on the one hand and with statistics on the other hand. Thus the triumvirate of my title: mathematics, numeracy, and statistics.

It is a lucky accident of the alphabet that numeracy fits between mathematics and statistics, since that is also how it fits in the curriculum. (So too does quantitative literacy, for that matter, a term that I use interchangeably with numeracy.) Much of numeracy is about mathematics, about relations embedded in real contexts, about mental estimation and logical thinking, about modeling and solving problems. But much of it is also about statistics, about seeking patterns in data and drawing appropriate inferences, about understanding randomness and estimating risks. Numeracy is not, however, the intersection of mathematics and statistics; neither is it their union. It is something different that sits halfway in between, sharing aspects of both but contributing elements that are distinctively its own.

What Are We Talking About?

A century ago, most students needed only a few years of education. At that time, secondary education was advanced education. But in recent decades, secondary schools have become the transition from elementary school to college. In particular, high school courses in algebra, geometry, trigonometry, analytic geometry, and calculus now offer a well paved and heavily traveled highway that leads increasing numbers of students directly from arithmetic to higher mathematics.

However, at the same time as secondary schools took on this pre-college mission, mathematics itself was expanding into a collection of mathematical sciences that now includes, in addition to traditional pure and applied mathematics, subjects such as statistics, financial mathematics, theoretical computer science, operations research (the science of optimization), and more recently financial mathematics and bioinformatics. (It is a little appreciated fact that most of the advances—and fortunes—being made in genetics, investment banking, and technology all derive from clever applications of sophisticated mathematics.) Although these mathematical sciences share with mathematics many foundational tools, each has its own distinctive character, methodologies, standards, and accomplishments.

Of these several mathematical sciences, the one that ordinary individuals most often encounter is statistics, originally meaning the science of the state. Created in the Napoleonic era when central governments began using data about population, trade, and taxes to assert control over distant territory, the value of systematic interpretation of data quickly spread to agriculture, medicine, economics, and politics. No longer just the science of the state, statistics now underlies not only economic reports and censuses, but also every clinical trial and opinion survey in modern society.

Yet school mathematics courses are still as they were before the era of statistics. They serve primarily to prepare students for traditional calculus-based college courses in mathematics, science, and engineering. High school mathematics devotes relatively little emphasis to topics designed to build a numbers-based bridge from the arithmetic of the elementary grades to the subtle and fascinating world of data and statistics.

Recognizing this neglect, about ten years ago the American Statistical Association (ASA) and the National Council of Teachers of Mathematics (NCTM) created a joint campaign in the United States designed to infuse more exploratory data analysis and elementary statistics into school curricula. Interestingly, they called this effort not "statistics" but "quantitative literacy" to avoid the negative public image that required college statistics courses had created in the minds of many graduates, notably educators.

Meanwhile, NCTM adopted the term "data analysis" to refer to the elementary parts of statistics that are included in its standards for school mathematics. So for example, whereas in a college setting one might expect to see the phrase "statistics and probability," in NCTM documents one generally encounters "data analysis and probability."

Thus four different terms in widespread use—quantitative literacy, statistics, numeracy, and data analysis—appear to have overlapping (and therefore uncertain) meanings. To add to the confusion, many mathematicians distinguish between the term "quantitative literacy" and "mathematical literacy." They reserve the former for broad goals for all citizens, the latter for specific skills needed by students who will use the tools of traditional mathematics in their future careers or further study. Mathematicians are inclined to think that mathematical literacy is what they teach, and that everything else that makes up quantitative literacy are other teachers' responsibilities.

Notwithstanding the lack of agreement on the precise meaning of either term, I use "numeracy" and "quantitative literacy" interchangeably, not wishing to get caught up in a definitional discussion that can best be described as arbitrary. However, I do generally distinguish between mathematics and statistics—since they are, after all, very different subjects—and between either of them and the subject of this talk: numeracy. My goal today is to help clarify the distinction between numeracy and the better-known disciplines of mathematics and statistics.

Taking Data Seriously

I begin with a statement of utmost simplicity. As physics can be described as the science of energy and biology as the science of life, so mathematics can be thought of as the science of patterns and statistics-in its modern form-as the science of data. This fundamental observation provides a very useful perspective from which to think about the relative merits of mathematics, numeracy, and statistics.

Despite its occasional use as a euphemism for statistics in school curricula, quantitative literacy (numeracy) is not the same as statistics. Neither is it the same as mathematics, nor is it (as some fear) watered-down mathematics. In fact, quantitative literacy is not really a science at all; it is more a habit of mind, an approach to problems that employs and enhances both statistics (the science of data) and mathematics (the science of patterns).

Mathematics teachers often resist emphasizing data since the subject they are trying to teach is about Platonic ideals—numbers and functions, circles and triangles, sets and relationships—not messy, unpredictable real-world data. Recognizing this difference helps explain why mathematicians are wont to distinguish between "mathematical literacy" and quantitative literacy": this kind of distinction helps fence off mathematics proper from incursions by the purveyors of data.

However, employers and members of the public are often frustrated by this "elitist" stance of mathematicians since school graduates so often seem inexperienced in dealing with real data. After all, the real world presents itself more often in terms of data than in the Platonic idealizations of mathematics. Moreover, statisticians wince whenever mathematicians deign to teach statistics since mathematicians rarely honor the data for what they are, but usually try to force them as quickly as possible into a predetermined (Platonic) distribution from which general conclusions can more readily be drawn.

As statistics differs from mathematics, so numeracy differs from both mathematics and statistics. Unlike statistics which is primarily about uncertainty, numeracy is often about the logic of certainty, especially deductions and calculations. Yet unlike mathematics which is primarily about a platonic realm of abstract structures, numeracy is often anchored in data derived from and attached to the empirical world. Surprisingly to some, this inextricable link to reality makes quantitative reasoning every bit as challenging and rigorous as mathematical reasoning. (Indeed, evidence from the Advanced Placement exams in the United States suggests that students of comparable ability find data-based statistical reasoning more much difficult than symbol-based mathematical reasoning.)

Taking data seriously has implications not only for topics such as number, measurement, and statistics, but also for the bedrock of high school mathematics: the study of functions. For example, despite the complexity of its algebraic formula, the normal distribution is as ubiquitous as linear and exponential functions. As citizens, it is very helpful to understand that repeated measurements (of the same thing) as well as multiple measurements (of different although similar things) tend to follow the normal distribution. Knowing why some distributions (e.g., salaries, size of cities) do not follow this pattern is equally important, as is understanding something about the small size of the tails of the normal distribution (which can be very helpful in thinking about risks). Employers would far rather that students understand the significance of three- and four-sigma variations from the mean than that they can calculate accurately the impact of a projectile launched at a 45 degree angle—even though the latter is far more likely than the former to be found in a school mathematics course.

It is interesting to ponder for a moment why quadratic and periodic functions are studied extensively in school whereas the Gaussian normal distribution is all but ignored in high school mathematics. One reason is the historically close affiliation of mathematics with physics: the functions most prominent in school mathematics are those most used to model common physical phenomena (gravity, motion). A second reason is that the mathematical characteristics (area, slope) of these functions can be calculated in terms of elementary functions, generally of the same family. And a third is the association of the normal distribution with statistics, a topic that, as we have seen, mathematicians have tended to shun.

In a numeracy-centered curriculum, however, students would learn early and often, and in many different contexts, the features and significance of the more common and powerful parts of mathematics such as the normal distribution. Despite what mathematicians implicitly assume (and sometimes explicitly argue), it is preposterous to expect students to wait to learn things until they know enough to learn them rigorously. If we followed that guideline for science, none of us would know anything at all about DNA or black holes—since none of us knows these things rigorously. The content and order of the curriculum should be dictated first and foremost by the value of topics for students' lives.

Illustrations of Numeracy

With utility rather than tradition as a guide, I offer a rather lengthy list of useful aspects of numeracy that are widely expected of high school graduates, aspects that teachers rarely teach and that colleges rarely emphasize in the courses they offer to prospective teachers. All these topics are legitimate parts of quantitative literacy—important skills that would enhance any person's life and work, skills rich in mathematics and statistics but neglected by traditional mathematics curricula:

About numbers and measurement:

- *Measurement*. Direct and indirect measurement. Use of appropriate instruments (rulers, tapes, micrometers, pacing, electronic gauges, plumb lines). Squaring corners and constructions. Estimating tolerances; detecting and correcting misalignments.
- *Calculation*. Strategies for checking reasonableness and accuracy. Significant digits; interval arithmetic; errors and tolerances. Spreadsheet methods for handling problems with lots of data.
- *Mental Estimation*. Estimating orders of magnitude. Quick approximations of total costs, distances, times. Proportional reasoning. Mental checking of calculator and computer results.
- *Numbers*. Scientific notation; units and conversions. Intuitive comprehension of extreme numbers (lottery chances, astronomical distance). Decimal, binary, octal, and hex coding; ASCII code; check digits.
- *Index Numbers*. Creation of stock market averages; consumer price index; gross national product; unemployment rates. Definitions and deficiencies; uses and abuses.

About space and geometry:

- *Dimensions*. Geometric dimension (linear, square, and cubic) vs. coordinate dimensions in multivariable phenomena. Proper vs. improper analogies. Discrete vs. continuous dimensions.
- *Dimensional Scaling*. Relation of linear, area, and volume measures under proportional scaling; fractal dimensions.
- *Spatial Geometry*. Shapes in space; interpreting construction diagrams. Calculating angles in three-dimensions (e.g., meeting of roof trusses); building three-dimensional objects and drawing two-dimensional diagrams.

- *Global Positioning*: Map projections, latitude and longitude, global positioning systems (GPS); local, regional, and global coordinate systems.

About data and risk:

- *Financial Mathematics*. Loans, annuities, insurance. Personal finance; nonlinear impact of changes in interest rates. Investment instruments (stocks, mortgages, bonds).
- *Data Analysis*. Visual displays of data (pie charts, scatter plots, bar graphs, box and whisker charts). Quality control charts. Recognizing and dealing with outliers.
- *Risk Analysis*. Estimates of common risks (e.g., accidents, diseases, causes of death, lotteries). Confounding factors. Communicating and interpreting risk.
- *Probability*. Chance and randomness; hot streaks; bias paradoxes.

About planning and modeling:

- *Planning*. Allocating resources; preparing budgets; determining fair division; negotiating differences; scheduling processes, decision trees; systems thinking.
- *Growth and Variation*. Linear, exponential, quadratic, harmonic, and normal curve patterns. Examples of situations that fit these patterns (bacterial growth, length of day) and of those that do not (e.g., height vs. weight; income distribution).
- *Mathematical Modeling*. Abstracting from real-world situations; reasoning within mathematical models; testing results for suitability and accuracy; revision and repetition of modeling cycle.
- *Information Systems*. Collecting and organizing data; geographic information systems (GIS) and management information systems (MIS); visual representation of data.
- *Scientific Modeling*. Common mathematical models such as acceleration, astronomical geometry, electrical current, genetic coding, harmonic motion, heredity, stoichiometry.
- *Technological Tools*. Facility with scientific and graphing calculators, spreadsheets, statistical packages, presentation software, and Internet resources. Experience converting data from one form and system to another.

About reasoning and inference:

- *Statistical Inference*. Rationale for random samples; double blind experiments; surveys and polls; confidence intervals. Causality vs. correlation.
- *Scientific Inference*. Gathering data; detecting patterns, making conjectures; testing conjectures; drawing inferences. Verifying vs. falsifying theories.
- *Verification*. Levels of convincing argument. Legal reasoning ("beyond reasonable doubt" vs. "preponderance of evidence"). Informal inference (suspicion, experience, likelihood). Logical deduction;
- *Mathematical Inference*. Assumptions, conclusions, and counterexamples. Axiomatic systems; logical deduction; theorems and proofs. Mathematical "induction."

Making Mathematics Meaningful

Connecting mathematics to authentic contexts is one way to make mathematics meaningful, but it demands delicate balance. On the one hand, contextual details camouflage broad patterns that are the essence of mathematics; on the other hand, these same details offer associations that are critically important for many students' long-term learning. Few can doubt that the tradition of decontextualized mathematics instruction has failed many students who leave high school with neither the numeracy skills nor quantitative confidence required for today's society. This tradition of using mathematics as a filter for future academic performance is reinforced by increasing pressure for admission to colleges and universities. These pressures skew school curricula in directions that are difficult to justify since they leave many students functionally innumerate.

Whereas the mathematics curriculum has historically focused on school-based knowledge, quantitative literacy involves mathematics acting in the world. Typical numeracy challenges involve real data and uncertain procedures, but require primarily elementary mathematics. In contrast, typical school mathematics problems involve simplified numbers and straightforward procedures, but require sophisticated abstract concepts. Effective quantitative instruction requires a good supply of mathematically rich tasks that are authentic, intricate, interesting, and powerful:

Authentic:
- Portray common contexts and honest problems.
- Employ realistic data, often incomplete or inconsistent.
- Meet expectations of employers and other users of mathematics.
- Use realistic input and output; avoid artificial worksheets.
- Reflect the integrity of both mathematics and the domain of application.

Intricate:
- Expect students to identify the right questions to ask.
- Require more than substitution into formulas.
- Employ multi-step procedures.
- Stimulate thinking that is cognitively complex.
- Confront students with incomplete (or inconsistent) information.
- Demonstrate the value of teamwork.

Interesting:
- Offer multiple means of approach.
- Touch on areas of interest to students.
- Appeal to a large number of students.
- Invite many variations and extensions.
- Provide horizontal linkages to diverse areas of life and work.

Powerful:
- Connect graphical, numerical, symbolic, verbal, and technological approaches.
- Offer vertical integration from elementary ideas to advanced topics.
- Propel students to more advanced mathematics.
- Expand students' views of mathematics and its potential.
- Demonstrate the value of mathematics to the modern high performance workplace.

Educators know all too well the common phenomenon of compartmentalization, where skills or ideas learned in one class are totally forgotten when they arise in a different context. This is an especially acute problem for school mathematics where the disconnect from meaningful contexts creates in many students a stunning absence of common number sense. To be useful for the student, numeracy needs to be learned and used in multiple contexts—in history and geography, in economics and biology, in agriculture and culinary arts. Numeracy is not just one among many subjects but an integral part of all subjects.

Who Owns School Mathematics?

These ideas, if placed on the table alongside your own secondary school curriculum, or the traditional U.S. curriculum (elementary algebra, geometry, intermediate algebra, trigonometry, analytic geometry, pre-calculus, calculus), or the new NCTM standards (number and operations, algebra, geometry, measurement, data analysis and probability, problem solving, reasoning and proof, communication, connections, and representation) raise a central question of importance to us all: just who really owns school mathematics? Who gets to decide what students should be taught, how they should be taught, and what they should be expected to know and be able to do? Do mathematicians or mathematics teachers have any greater standing in this decision than parents or taxpayers or school administrators? In many countries, my own included, this is a very contentious issue, pitting governments against educators and scientists against teachers.

Although I can really speak only about what I see and hear in the United States, I suspect you will recognize many of these arguments from your own experience. Reformers typically want school graduates who can solve problems, communicate mathematically, reason mathematically and appreciate the value of mathematics. In contrast, parents want their children to learn basic skills, especially the same mathematics with the same procedures that they studied when they were in school. In addition, parents want their children to learn whatever is needed to get good scores on standardized school and college admission tests.

Politicians generally want school graduates to know and be able to use basic skills, especially those that are essential for work or further education. They also would like students to do well on comparative international tests, and they pay lip service to wanting citizens who can understand quantitative aspects of public policy issues. (This cuts two ways, however, since informed citizens can more readily see through the gloss that is typical of politicians' prose.)

Employers tend to be more practical-minded. They want employees who can apply arithmetic and geometry; use formulas; understand elementary statistics; read and interpret charts, graphs, tables, and instruments; use computers intelligently; and recognize mathematical issues hidden in work situations. Scientists, too, want school graduates who can apply mathematics in real-world contexts, especially those in which the mathematics is implicit rather than explicit (as it generally is in school mathematics courses). Like mathematicians, scientists expect a high degree of number and symbol sense as well as the ability to use a broad repertoire of basic tools from algebra and geometry. In addition, mathematicians—but hardly anyone else—hope that students understand the importance of logical proof in mathematics and are able to work standard textbook mathematics problems.

Where does this leave teachers? Right in the middle of warring parties. Mathematics teachers certainly want their students to master basic skills, especially those needed to do well on standardized tests. They also all want their students to be able to solve problems, to understand rather than memorize, to think mathematically, and to know enough algebra and geometry to be able to move on to higher mathematics in college or technical schools. But for teachers to achieve any of these objectives, the various prospective owners of school mathematics must first sit around the table and hammer out an agreement on goals.

This won't be easy. In both of our countries, and in many others, the record of the traditional mathematics curriculum has not been especially strong. It has satisfied neither employers nor parents, neither scientists nor politicians. It has left most student quantitatively unprepared for the future. Yet it has the weight of tradition on its side, together with all the infrastructure of textbooks, tests, and teacher training that go with it.

I have argued for the value and efficacy of numeracy (or quantitative literacy) as an alternative to traditional approaches. But numeracy is not now part of the curriculum—mathematics is. So for any educational jurisdiction to move in this direction, they need to face forthrightly some very fundamental questions about mathematics:

- Why is mathematics required in school?
- How much high school mathematics can all students be expected to learn?
- Is understanding a realistic (or even desirable) goal of school mathematics?
- How important are proofs for school mathematics?
- How important is it that problems be set in authentic contexts?
- How many mathematics are there?

This last question is perhaps the most important, and most intriguing. (I leave it to you to tease out its full meaning.) I suggest that candid answers to these questions will inevitably tilt the balance in favor of numeracy.