Quantitative Literacy and School Mathematics: Percentages and Fractions

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The secondary school mathematics curriculum emphasizes algebra as a necessary preparation for college calculus and statistics. Approximately 40% of college graduates, however, are in non-quantitative majors that do not require calculus or statistics and have little need for algebra beyond proportional and linear reasoning. Yet the elementary school curriculum presents common fractions symbolically as an introduction to high school algebra. This approach may "turn off" some very bright students, both those who might otherwise be interested in careers in science, technology, engineering and mathematics (STEM) and those who may graduate from college with majors such as journalism or political science that do not require much mathematics. Even non-STEM students need to be quantitatively literate to excel in their fields and to be capable citizens in a modern data-based democracy where most social and political issues involve quantitative reasoning.

To increase the effectiveness of quantitative literacy throughout the school curriculum, this paper explores the possibility of delaying, minimizing, or eliminating the manipulation of common fractions as mathematical objects and

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of replacing it with a more applied study of fractions in the context of percentages and rates.

- A greater focus on percentages and rates could enhance the quantitative literacy of all students and improve the motivational support provided by parents and teachers while still introducing important topics such as scaling, conversion, changing units, and symbolic notation.
- A greater focus on the ordinary English grammar involved in communicating about rates and percentages would allow students to become better consumers of information presented in tables and graphs.
- A greater focus on the uses of ratios would allow educators to prepare students for more advanced topics such as standardizing and Simpson's paradox that are common in everyday media but rarely covered in the current school curriculum.

This paper also discusses the possibility of introducing these quantitative literacy topics either as a pre-algebra bridging course or as a quantitative or statistical literacy course in place of algebra II for those students interested in non-quantitative majors in college.

Goals of mathematics education

The National Council of Teachers of Mathematics (NCTM), the principal professional society for K–12 mathematics education, says that its goal is to ensure "mathematical learning of highest quality for all students" (NCTM, 2007). This broad goal leaves open the choice of topics students should learn and the order in which they should be learned. In practice, it appears that elementary school mathematics prepares school children for high school mathematics, which in turn prepares students for college mathematics. Thus, the choice and order of topics at the school level may be influenced—if not driven—by the mathematical needs of students at college.

The mathematical needs of college students can be inferred from the mathematics and statistics courses they take—data that is gathered regularly by a survey by the Conference Board of the Mathematical Sciences (CBMS). According to the 2000 CBMS survey (Lutzer, D, J., J. W. Maxwell, & Rodi, 2002) of U.S. four-year colleges during the 2000 fall semester:

- 217,000 students took remedial mathematics [General Mathematics (30,000), Elementary Algebra (70,000) and Intermediate Algebra (117,000)];
- 723,000 took introductory pre-calculus mathematics [College Algebra

(211,000), Elementary Functions (105,000), Mathematics for Liberal Arts (86,000), Finite Mathematics (82,000) and Mathematics for Elementary School Teachers (68,000)];

- 297,000 took Calculus I; and
- 155,000 took Elementary Statistics in mathematics/statistics departments.

The courses most frequently mentioned by departments of mathematics as one of the top three courses taken by K–3 education majors were a multi-term mathematics course designed for elementary education majors (48%), followed by College Algebra (42%), Mathematics for Liberal Arts (39%), a single-term mathematics course designed for elementary education students (32%), and statistics (29%). The courses most frequently mentioned by departments of statistics as one of the top three statistics courses taken by K-3 education majors were elementary statistics (63%), statistical literacy (33%), and a single-term statistics course for elementary education majors (26%).

While the CBMS survey is the most accurate survey available of mathematics and statistics courses taken by U.S. college students, it has three limitations. First, it is a fall-only survey, so courses that are taught year-round (e.g., college algebra and statistics) may have different year-round enrollments than those taught primarily in the fall (e.g., Calculus I). Second, it does not count those students or courses taught outside mathematics and statistics departments. This is a problem for statistics at four-year colleges since statistics is often also taught in other departments such as business, psychology, and sociology. Or as the CBMS report noted, "in fall 2000 there were fewer than 100 statistics departments in the U.S., and almost 1,400 mathematics departments. Consequently the numbers reported by statistics departments would not include the students from the vast majority of colleges." Third, there is no way to link courses with students—we cannot calculate what percentage of college graduates take a given course as their last course in mathematics. For example, college algebra may be taken as a prerequisite for calculus by some and as a terminal course for others.

These three limitations of the CBMS survey are important in identifying the reasons that college students take a mathematics course. Is it because of general education requirements, the requirements of their majors, their personal interests, or their need for remedial courses as a prerequisite for any of the foregoing?

An alternate approach to identifying the mathematical needs of college students is to examine bachelor's degrees earned by major. In 2003, there were 1.35 million bachelor's degrees awarded at U.S. four-year colleges and universities (U.S. Census Bureau, 2006, Table 289). If we assume that calculus was taken by all students graduating in science, technology, engineering and mathematics (STEM), then 12% of college graduates were required to take calculus. If we assume that statistics was taken by all students graduating in business, the social sciences, psychology, health sciences and biology, then 48% of college graduates were required to take statistics. Even though some students may take calculus or statistics even if not required by their major, and some majors may require calculus as a prerequisite for statistics, this leaves approximately 40% of college graduates with majors that do not generally require a specific mathematics course. These non-quantitative majors include education, visual and performing arts, communication and journalism and English, as well as the liberal arts, humanities, general studies and interdisciplinary studies. Many of these students must take one or more college mathematics courses as part of their general education requirements. But the lack of a specific mathematics requirement may tell students in these non-quantitative majors that their major department sees no direct benefit of mathematics for their major.

Even the role of mathematics in general education is changing. At some schools, college algebra no longer satisfies a quantitative general education requirement. For example, Arizona State University recently removed college algebra from the list of courses students can use to fulfill the numeracy requirements for general studies. "The department has taken this action because it believes students requiring only one mathematics course in their college experience should be introduced to mathematics that is more applied in nature. We further believe any student taking college algebra should have every intention of taking another mathematics course" (Isom, 2004). Briggs (2006) reviews the "algebra dilemma" in designing a successful liberal arts mathematics course and argues that "less could be better." He suggests that we should "avoid doing algebra when there is no ulterior purpose and let the applications determine the necessary mathematics."

Unfortunately, there is no summary of the mathematics courses required for general education at U.S. colleges and universities. Courses such as college algebra, statistics, mathematics for liberal arts, quantitative literacy and statistical literacy are often used for this purpose along with courses designated as satisfying a quantitative reasoning requirement. For a review of the topics commonly found in quantitative literacy courses, see Gillman (2006) for a mathematics-centered review and Madison (2006) and Schield (2004a) for a broader view.

Overlaid on the issue of students' mathematical needs is the issue of attitudes towards mathematics. All too many students have a negative attitude toward mathematics. The Third International Mathematics and Science Study

(TIMSS, 1999, Exhibit 4.10) found that 35% of the U.S. 8th graders surveyed say they have a positive attitude toward mathematics, 50% say their attitude is between negative and positive (neutral) toward mathematics and 15% say they have a negative attitude. For the girls in this study, the percentages were 32%, 52% and 16%, respectively. Since these 8th graders typically have not yet had algebra or geometry, a relevant explanation may be the teaching of fractions.

Mathematics for the other 40%

School mathematics has many goals, one of which is to provide students with the mathematical concepts and training they need to function as quantitatively literate citizens in a modern democracy. Training students to study mathematics or science in college is another goal. But if this STEM goal conflicts with the quantitative literacy goal, the algebra-centered school mathematics curriculum may become dysfunctional. It may inadvertently encourage bright college bound—but non-mathematics oriented—students to avoid quantitative thinking even when it is appropriate and important.

The 40% of college graduates with non-quantitative majors are more likely to become elementary school teachers, journalists, lawyers, policy makers and religious, social and political leaders. These are the students who are likely to take courses with titles like Mathematics for Liberal Arts and Statistical Literacy. These liberal arts majors may not need an algebra-centered curriculum to help them reason quantitatively, that is, to form sound arguments and make informed decisions about matters for which numerical evidence is offered. But all too many humanities majors are innumerate, or quantitatively illiterate. Among other deficiencies, they have surprising difficulty reading tables of percentages (Atkinson and Wills, 2007) and Schield (2006a).

Addressing this conflict does not mean supporting a watered-down mathematics curriculum for potential STEM students, rather, just the opposite. A slightly different approach to the teaching of fractions can teach quantitative relationships that are useful to non-STEM students, that can challenge STEM students, and that might even attract bright non-STEM students into STEM majors in college. A first step along this road is to identify the quantitative literacy needs of college-educated citizens, in particular, of students in the arts and humanities (e.g., English, history, philosophy, aesthetics, and political science).

Student weaknesses related to mathematics

Lutsky (2006) analyzed writing portfolios of 200 students at Carleton College, a highly selective liberal arts college. He found that a third of these college students failed to use quantitative reasoning (QR) when it should have been *central* to their analysis, and nine in ten failed to use quantitative reasoning when it was *peripheral* but of potential benefit to their argument.

An earlier analysis prepared for the International Association for Statistical Educators identified several categories of problems involving quantitative or statistical literacy (Schield, 2004b) as follows.

Problems comparing counts or amounts using ordinary English: Students know that "8 is 6 more than 2" and that "8 is 4 times [as much as] 2." But they may mistakenly think "8 is 400% more than 2." They are quite comfortable—but mistaken—in saying "2 is 4 times less than 8." They are amazed that 15% is 50% (but not 5%) more than 10%.When told that "Jane is half as old as Tom; Tom is twice as old as Mary" and asked if Jane and Mary are the same age, their answer, "Yes," is correct. When told that "Jane is 50% younger than Tom; Tom is 50% older than Mary" and asked if Jane and Mary are the same age, their answer, "Yes," is incorrect.

Problems describing percentages and rates in ordinary English: Percentage and rates are common in graphs, yet one study found that one college student in five could not correctly read the simple pie chart of percentages shown in Figure 1 (Schield, 2006a). Percentages were featured in 70% of the graphs in *USA Today On-Line Snapshots* (Schield 2006c), yet many students were unable to properly interpret their meaning as evidenced by Figures 2–5. In reading Figure 2, students mistakenly concluded that 24% of all adults have two dogs—rather than 24% of all dog owners have two dogs.

In Figure 3, some students thought the bar graph was wrong since the percentages add to more than 100%—not realizing that the alternatives were non-exclusive in the survey. In Figure 4, some students mistakenly concluded that 43% of the happy people surveyed are married rather than that 43% of the married people surveyed are happy.

In Figure 5, the percentages add to 92% and the age groups are exclusive (but not exhaustive), so students cannot tell whether 29% of those ages 21–25 received a DUI or 29% of those receiving a DUI are ages 21–25. In Figure 6,

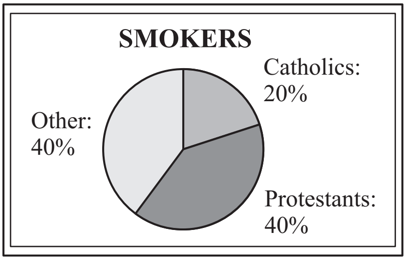

Figure 1. Pie Chart

Figure 2. Bar Chart (Sum = 100%)

Figure 3. Bar Chart (Sum > 100%)

Figure 4. Bar Chart (Sum < 100%)

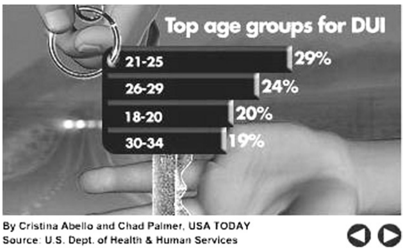

Figure 5. Bar Chart (Sum ~ 100%)

Figure 6. Bar Chart (Sum ~ 100%)

the percentages add to 98% and the income groups are exclusive and exhaustive so students cannot tell if 15% of guests from low-income households bring gifts or if 15% of guests who bring gifts are from low-income households.

Similar weaknesses are apparent in reading tables of rates and percentages. In reading Table 1, 19% of students surveyed mistakenly thought the circled 25% said that 25% of females are blacks rather than 25% of blacks are female (Schield, 2006a). In reading Table 2, among those surveyed, 55% of students, 53% of professional data analysts and 30% of college faculty mistakenly thought the circled 20% said that 20% of runners are female smokers (or did not know) rather than 20% of female smokers are runners. These error rates are important since percentages and rates were featured in 40% of the tables in the 1997 U. S. Statistical Abstract.

College students also have considerable difficulty determining part and whole in ratios presented in tables and graphs. In reading Table 3, students

	SEX		
RACE	Male	Female	TOTAL
Black	75%	25%	100%
White	50%	50%	100%
Other	40%	60%	100%
TOTAL	50%	50%	100%

Table 1. 10% Row Table

PERCENTAGE WHO ARE RUNNERS			
	Non-smoker	Smoker	Total
Female	50%	20%	40%
Male	25%	10%	20%
Total	37%	15%	30%

Table 2. Two-way half table

were asked to describe the 59.3% in the cell for black males. About a third of the students mistakenly concluded that 59.3% of overweight or obese adults were black males rather than that 59.3% of black male adults are overweight or obese. They seemingly ignored that this incorrect statement was highly disproportional since less than 7% of U.S. adults are black males. They ignored the fact that the table Totals are averages—not sums.

Adults	Men	Women	Total
Total	53.5	40.5	48
Whites	52.9	39.7	46.1
Blacks	59.3	49.6	54.4
Other	51.4	40.8	46

Table 3. Percentage overweight or obese

Augsburg College students studying statistical literacy have difficulty interpreting percentages when they are expressed in ordinary English. They are not sure if "the percentage of men who are runners" is the same as "the percentage of men among runners." When given "20% of men who run are smokers," they often conclude that "20% is the percentage of men who run who are smokers." They cannot see a difference between "the percentage of male runners who smoke" and "the percentage of men who run and smoke." They are exposed to sports grammar (e.g., "percentage of passes completed" or "percentage of completed passes") where there is a natural whole so the syntax is irrelevant and both have the same meaning. Without training, they don't see that "the percentage of male smokers" has no natural whole and could be "the percentage of males who are smokers" or "the percentage of smokers who are male."

Problems comparing percentages and rates using ordinary English: A study involving Figure 1 dealing with the incidence of smoking in relation to Protestant/Catholic identity found that 60% of students and data analysts surveyed mistakenly concluded from this table that "Protestants (40%) are twice as likely to be smokers as are Catholics (20%)" (Schield, 2006a). A correct statement would be, "Protestants (40%) are twice as likely among smokers as are Catholics." The comparison of ratios, rates and percentages in ordinary language requires using English in a very precise manner. Small changes in syntax can produce large changes in semantics.

Problems involving weighted averages of measurements or percentages: Steen (2001, pp. 11) described "understanding the behavior of weighted averages used in ranking colleges, cities, products, investments and sports teams" as a key topic in quantitative literacy for citizenship. Students in non-quantitative majors may not realize that many of the statistics we read are not simple averages—they are weighted averages where the average depends on the number in each component of the mixture.

College students have trouble comparing weighted averages for two groups with different mixtures in their respective populations. Supposing we find that the average weight of college seniors at St Thomas is 30 pounds more than the seniors at St. Catherine's. This could be an indication of overweight or obesity at St. Thomas. But students at St Catherine's college are mainly women while those at St. Thomas are equally split. The 30 pound difference could reflect different mixtures, that is, the difference in the number of men and women at the two colleges.

College students have even more difficulty with weighted averages when they involve two groups with different mixtures and when both the outcome and the mixture are expressed as percentages. Suppose the percentage of college students who go on to graduate school is much higher at St. Thomas than at Augsburg. This might reflect a difference in the quality of the education. But it might reflect a difference in the mixture of students. Suppose that children of college-educated parents are more likely to go to graduate school than children whose parents are not college educated. Suppose that college-educated parents are more prevalent among students at St. Thomas than among students at Augsburg. The failure to take into account the influence of this third factor—college educated parents—can confound the association between the two colleges and the percentage of their graduates who go on to graduate school.

Figure 7 illustrates a graphical technique for illustrating and standardizing weighted averages.

Consider the death rates at two hospitals: City and Rural. The overall death rate at each depends on the death rates for the two groups of patients:

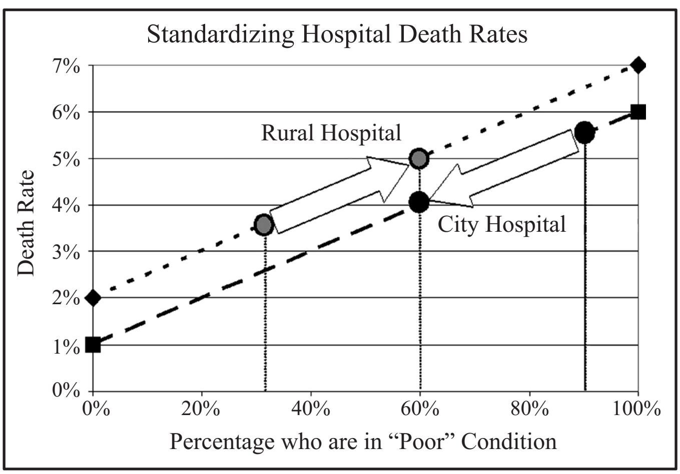

Figure 7. Standardizing Hospital Death Rates

those in poor condition (right side) and those in good condition: those not in poor condition (left side). The overall average is a weighted average—the average of the death rate for the two groups weighted by their prevalence: the percentage of patients who are in poor condition (horizontal axis).

In this case, the overall death rate is higher at a City Hospital (5.5%) than at a Rural Hospital (3.5%). Obviously patients in poor condition (right side) are much more likely to die than those in good condition (left side). Patient condition could be confounded with the hospitals and thereby influence the observed association between hospitals and death rates. Suppose that patients in poor condition are much more prevalent among patients at City Hospital (90%) than among those at Rural Hospital (30%).

When given the death rates for patients in poor and good condition for each hospital, students can standardize the prevalence of the confounder to the overall average (60%) and see that in this case the standardized death rate is reversed. The standardized death rate—the death rate obtained after taking into account the influence of a related confounder of patient condition—is now higher at Rural Hospital (5%) than at City Hospital (4%). This reversal is an example of Simpson's Paradox—a phenomenon that is all too common in everyday comparisons of averages, rates and percentages. This simple graphical technique illustrates how one can take into account the influence of a third factor on an average, a rate or a percentage.

More examples can be found in Schield (2006b) and Terwilliger and Schield (2004). Lesser (2001) provides a comprehensive review of weighted averages—the basis of Simpson's paradox.

Problems concerning student attitudes. It may seem inappropriate to include attitudes when determining content, especially among primary school students, but by secondary school, if not by middle school, student attitudes affect student choices and performance.

Business majors at Augsburg College in spring 2003 were surveyed by the author on their major within business and on their attitude toward mathematics. Majors were classified in two groups: non-quantitative (management, international business, management information systems and marketing) or quantitative (accounting and finance). Attitudes toward mathematics were classified into two groups: "like math" (strongly like or like) and "dislike math" (neutral, dislike or strongly dislike). The result: 30% of quantitative majors and 70% of non-quantitative majors "dislike" mathematics. Almost 60% (40%/70%) of the students in management, international business, MIS or marketing are attributable to their dislike of mathematics. Note that this association does not say that their attitude toward mathematics *caused* students to choose non-quantitative majors, but the association suggests and supports this claim.

Attitudes are important in another way. Schau (2003) noted, "Many of us believe that attitudes impact students' achievement, [their] course completion, [their] future course enrollment, and [their] statistical thinking outside of the classroom." The less value students see in what they are learning, the less motivated they are to participate, to learn, to remember what they learned, and to use what they learned.

Interestingly, Schau found that college students see less value in studying statistics after completing the introductory research-oriented statistics course than they did before taking the course. It may be that high school students see less value in mathematics after taking algebra than they did before. It may be that grade school students see less value in mathematics after studying fractions than they did before. If this were so, it would represent a serious problem—even if immaturity is the underlying cause. Student attitudes affect their willingness to take further courses in a subject. Those grade-school children who see less value in mathematics are less likely to take the next courses in mathematics or, if required to do so, their performance may reflect their negative attitude. In the end, they may be far less likely to pursue STEM majors in college.

School students may not like multiplication or division, and they may not see much value in these operations immediately. But their teachers, parents and older peers are generally united in claiming that these skills are important—even in the age of computers and calculators. But when students and their older peers see little value in a subject such as manipulating fractions, students may start to question whether their teachers really have their best interests at heart.

'Attitudes' includes the attitudes of teachers and parents, which may account for much—if not most—of the difference in academic performance among K-6 school children. If teachers do not see the value in material required for a majority of their students, this may affect their attitude: they will not be excited about and persuasive in teaching such material. If parents do not think their children should learn the material, if they cannot see that the material is useful or how their children will benefit, their negative attitude may influence children in ways teachers cannot overcome.

Teachers may have more influence on student learning than does the choice of topics in a curriculum and parents may have more influence on student learning than do teachers.

Call for change

The NCTM has done much to review the mathematics curriculum, but more can be done to improve the school curriculum for the 40% of college students in non-quantitative majors.

Ideally the mathematics curriculum should help each student use their mind—at their level of understanding—to understand the world in quantitative terms. Mathematics provides some simple quantitative devices for taking into account related factors. Two-group comparisons take into account the size of a related factor chosen as the basis for the comparison either as a difference, a ratio or a percentage difference. Rates and percentages take into account the size of the group. Two-group comparisons of rates and percentages take into account both the different sizes of the two groups and the ratio for the group chosen as the basis for the comparison. Standardizing takes into account the influence of a related factor. Taking into account the influence of a related factor is what links the mathematics of percentages, rates, comparisons and standardization to quantitative literacy with its focus on mathematics in context.

Here are some recommendations:

1. Emphasize ordinary English. Mathematics educators should consider how ordinary English can be used in preparing students for algebra. Some might argue that the words of ordinary English cannot substitute for symbolic algebra. Yet English can convey quantitative ideas. Most—if not all—arithmetic operations and algebraic relationships can be expressed in ordinary sentences. Ordinary English can be used to make quantitative statements that are clear and unambiguous. Everyday graphics (e.g., pie and bar charts) can display the semantics of percentages just as Venn diagrams display the overlap between two groups or variables.

Including a wider-variety of ordinary English forms in teaching mathematical relationships may help improve the attitudes of school teachers and parents. Parents and teachers may encourage students to work harder in mathematics if they understand the value of what is being taught.

2. Distinguish percentages from fractions. Mathematics educators might rethink the relation between the teaching of percentages and the teaching of fractions. Teaching the manipulation of common fractions that are ratios of integers can provide an introduction to algebra which in turn provides a basis for calculus and statistics. But do college students in non-quantitative majors need to manipulate common fractions? They certainly need to manipulate percentages. But are percentages fractions?

Mathematically, percentages are fractions with a denominator of 100. But operationally percentages are not common fractions. To add common integer-ratio fractions such as ½ and ¾, one must scale at least one of the fractions to give them a common denominator so they can be added. But percentages—by their very nature—all have the same denominator: 100. There is never any need to rescale a percent before adding or subtracting. Operationally, percentages are much closer to integers or decimal fractions than they are to common fractions.

Consider a well-known mistake involving fractions: 1/3 + 1/5 = 2/8. In making this mistake, students apply whole-number addition where it is not appropriate. But would students add 33% to 20% and get 25%? Not likely! The mistake with common fractions seldom occurs when the fractions have a common denominator.

3. Be aware of how students and adults—even very bright people—avoid common fractions. How do they do this when dealing with everyday units such as time, money, distance, weight and volume? One way is to shift to a smaller unit so the fraction becomes an integer. In this way, half an hour become 30 minutes, half a dollar becomes 50 cents, half a foot becomes 6 inches, half a pound becomes 8 ounces and a third of a tablespoon becomes a teaspoon. Percentages function in the same way: a tenth of a unit becomes 10 percent—where 'percent' (one-one hundredth) functions as the new smaller unit.

Of course one can always go smaller than the smallest common unit—be it a second, a cent, an inch, an ounce, a teaspoon or a percent. Are fractions required? Yes, but not as often and they may not be common fractions. We can use decimal fractions. *Mathematically, decimal fractions are a type of common fraction. Operationally, decimal fractions are closer to whole numbers than to common fractions.* Now this may be questionable if students mistakenly think 0.17 > 0.7 because 17 is bigger than 7. But if the arithmetic of decimal fractions is easier than the arithmetic of common fractions—easier for teachers to teach, for parents to support and for students to learn,—then this would support the claim that decimal fractions are closer—operationally—to whole numbers than to common fractions.

Learning to add fractions with different denominators may be a critical step in a child's understanding of rational numbers. But do students in the humanities or educated citizens in a modern democracy need to distinguish rational numbers from irrational numbers? Do they need to know how to divide one common fraction by another when the few times they encounter this, they can convert them both to decimal fractions and use integer arithmetic to calculate the result?

There are three distinct situations that arise in adding common fractions:

- Those having identical denominators (e.g., percents with a denominator of "100," and rates with a common basis). Fractions having identical denominators are added by adding their numerators just like whole numbers for the same denominator (the same unit fraction). So, ¼ + ¾ = 4/4 and 25% + 75% = 100%.

- Those having commensurate denominators (e.g., ½ + ¼). Those having commensurate denominators can be easily scaled so they have a common denominator. For example, consider $(a/b) + (c/d)$ where $d = k*b$ so $b = d/k$. Thus, $(a/b) + (c/d) = [(ak)/d)] + (c/d) = [(ak)+c]/d$. If we want to add a quarter and a half dollar, we exchange the half dollar for two quarters (divide the half dollar by ¼ to get two-fourths) and add the two quarters with the one quarter to get the total of three quarters. In shifting from dollars to quarters, it seems that students have difficulty seeing that multiplying the dollars by four is the same as dividing the dollars by ¼.
- Those having incommensurate denominators (e.g., 1/4 + 1/5). One can express the addition of incommensurate fractions as a result of a double scaling. Consider $(a/b) + (c/d)$. If b and d are incommensurate, then a simple scaling is to use their product. Scale (a/b) by multiplying and dividing by d; scale (c/d) by multiplying and dividing by b. This gives, $(ad)/(bd) + (cb)/(bd)$ which gives the well-known result $(ad+bc)/bd$.

4. Introduce arithmetic operations using percentages and rates in context. Mathematicians create mathematical objects by omitting context. Quantitative literacy focuses on mathematical objects in context where the context makes a mathematical difference. Thus, ratios, fractions and percents encountered at school are typically mathematical objects. But the fractions, ratios, percentages and rates encountered in everyday usage typically appear in context. Eight-tenths or 80% is a mathematical object. "The percentage of U.S. toys that are made in China is 80%" is a percentage: a fraction in context where the numbers refer to things in reality. Likewise, a ratio of two dimensionless numbers is a mathematical object. A ratio in context (e.g., 30 miles per U.S. gallon or 12.8 kilometers per liter) is a ratio in context. As a mathematical object, a rate is simply a ratio. But the term 'rate' in context can mean a rate per unit time (the number of births per year), a prevalence (the unemployment rate among U.S. blacks 18-24 in the civilian labor force who are not in college was 18.3% in 2005) (U.S. Census Bureau, 2007, Table 581) or an incidence (the death rate is 817 per 100,000 U.S. population in 2004 (U.S. Census Bureau, 2007, Table 109)).

Any two real numbers can be added, subtracted, multiplied or divided. But the results are not always meaningful or appropriate for numbers in context.

Once students have identified that a percentage or rate involving counts of real things that can be identified by their membership in a group (e.g., men or women), then students can determine whether two percentages or rates have common or distinct parts. If they have distinct parts within a common whole, then their sum can be meaningful—provided these parts are exclusive. But if they have common parts involving two distinct wholes, then adding them may be meaningless. As an example, consider this problem. Suppose a company has a 60% market-share in the eastern U.S. and a 70% market-share in the Western US. What is their market-share in the entire US? It cannot be 130%. Here is a case where the addition of fractions (6/10 + 7/10 = 13/10) is correct but meaningless. Students need to be taught when a sum of fractions in context is meaningful and when it is not.

Fractions in context have different forms and the context determines what can and cannot be done operationally. The operations that can be done or not done are not always consistent from context to context. This makes it imperative that educators help students interpret fractions in context in a sense-making way rather than in an abstract algorithmic way.

5. Be aware of objections to increasing the focus on percentages and rates in context. It is all too easy to say that just because we may not do something in everyday life, that we should not have to learn it. Students used to learn how to take a square root, but now calculators do that for us. Does this mean students should not have to learn how to divide or multiply or subtract or add since calculators can do this for us? Absolutely not! Calculators do not tell us how to enter the information. Calculators do not provide an estimate of the answer so we can see that we made a mistake in entering the problem. Calculators may not help us develop a conceptual understanding that is crucial to becoming educated.

Wu (2002) claims that "fractions hold the potential for being the best kind of 'pre-algebra." He noted that, "the subject of fraction arithmetic—usually addressed in grades 5 and 6—is rife with opportunities for getting students comfortable with the abstraction and generality expressed through symbolic notation." He illustrated this in adding two fractions, $(a/b) + (c/d) = (ad+bc)/(bd)$, and noted the truth of this equation holds regardless of whether the variables are whole numbers, fractions, finite decimals or polynomials [assuming non-zero denominators]. The same holds true when multiplying two fractions: $(a/b)(c/d) = (ac)/(bd)$. For Wu, "there is no generality or abstraction without symbolic notation."

6. Identify advantages to other mathematical topics that might be introduced to help students develop their conceptual powers instead of common fractions. Taking Wu's claims as true, one can still ask if tables, graphs and ordinary English statements are a form of symbolic notation. One can ask if there are other mathematical ideas that could introduce students to abstraction and symbolic notation. Ratios (including simple percentages and rates), linear models ($y = a + bx$), weighted averages ($z = [x$*weight of $x] + [y$*weight of $y]$) and scaling and proportional reasoning ($a/b = c/d$) can all be demonstrated using symbolic notation and are encountered more often in everyday life than adding common fractions.

Abstraction and generality are important in developing one's conceptual powers. Learning that the unit of measurement is a human choice is an important step in cognitive development. Are there ways to use fractions that are a more natural fit with the way people talk in everyday life?

Instead of using common fractions to introduce symbolic notation and the idea of scaling, one alternate might be to use everyday rates. Everyday rates are not mathematical rates: the slope of a line (y/x or dy/dx). Such a slope may be described as a 'grade' measured in percent.

Everyday rates are ratios of two related counts. They may be incidences per unit time such as birth rates and death rates, or prevalences at a moment in time such as the unemployment rate. Whereas percentages are all 'per 100,' these everyday rates specify the unit of measure. At first changing the scale seems no different with rates than with the everyday units of time. In the U.S., the birth rate in 2003 was 14 per 1,000 people (U.S. Statistical Abstract (2007, Table 76). Thus it is 140 per 10,000 people and 1,400 per 100,000 people.

But with real-world rates, the numerator and denominator are not necessarily two independent quantities: they are often linked. Men cannot give birth so the birth rate in the U.S. is 29 per 1,000 women—since women made up 51% of the population and 14/0.51 is 29. Women younger than 15 or older than 44 seldom give birth so the birth rate in the U.S. is 66 per 1,000 women aged 15–44—since 44% of all women are ages 15–44 and 29/0.44 is 66. (U.S. Statistical Abstract, 2007, Table 78). In these cases, the numerator in reality remains unchanged (U.S. births in 2003) and is independent of the size of the denominator (adults, women or women ages 15–44).

The linkage between numerator and denominator depends critically on the context. In a percentage, the numerator (the class counted) is always a subset of the denominator so changing the size or the group can change the numerator. The same is true for many rates. Thus, the accident rate per 100,000 licensed drivers is generally not the same as the accident rate per 100,000 cars or the accident rate per 100,000 bars. Recall the birth rates mentioned above. Women ages 18–19 are less than 10% of women age 15–44, but the birth rate for the younger group, 71 per 1,000 among women 18–19, is not ten times as high. Obviously the numerator—number of births among this age group—is less than the number of births among those women age 15–44. Using rates where the numerator and denominator are linked because of the context introduces a new factor that is not obvious in dealing with a common fraction a/b.

As another example, compare the accidental death rates between Arkansas and Hawaii in 1996. Arkansas has a higher accidental death rate than Hawaii (36 vs. 18) per 100,000 registered vehicles. But Hawaii has a higher accidental death rate than Arkansas (35 vs. 7) per 1,000 miles of road (U.S. Statistical Abstract, 1998, Tables 143, 1019 and 1029). Once again students see that the choice of the unit of measure not only changes the size of a statistic—it can influence the direction of an association. If students are to be statistically literate, they must understand that the numerator and denominator can be linked by the choice of the denominator—not just by the size of the unit in the denominator.

Scaling is mathematical. But if students are to be quantitatively literate, they need to learn that the choice of the group—the basis of a comparison—can strongly influence the size of a number or a statistic. Teaching fractions as having an independent numerator and denominator overlooks this dependency on context: a material element that is critical in the real world—and in the conditional probability of statistics.

It may be helpful for mathematical educators to use this focus on context to categorize the transformation of ratios. Consider three groups: numerator is directly proportional to denominator (shifting a rate from per 100 to per 1,000: mathematical scaling), numerator is independent of the denominator (shifting the birth rate from all people to just women), and the numerator is related to or dependent on the denominator but not directly proportional (shifting the accident rate denominator from registered vehicles to miles of road). By focusing just on the first group, students may have been denied access to more complex applications of mathematics that are relevant in everyday life.

Weighted averages provide another way to introduce abstraction. One wonders why the weighted average of counts in separate group, $[(a/b), (c/d)]$, is not included as $(a+c)/(b+d)$ since the weighted average is a real and valuable concept in everyday life. For example, if there are 30 smokers among 90 men and 5 smokers among 10 women, then there are a total of 35 smokers among these 100 individuals: 35% of these people are smokers. Note that the 35% is the average of the 33% among men and the 50% among women weighted for the mixture of men (90%) and women (10%): 0.9*33% + 0.1*50% = 35%.

In quantitative literacy, context counts. Even if ratios in context were inferior to mathematical objects such as common fractions in terms of introducing students to symbolic notation and abstraction, the benefits from a heightened focus on context along with improved teacher understanding and persuasiveness, from improved parental involvement and from increased student awareness of their benefits might more than compensate for their formal weaknesses having less emphasis on symbolic notation.

7. Identify places in the curriculum to introduce or embed the study of fractions in context. Eliminating the abstract algebra-like manipulation of common fractions in elementary schools may be overly drastic at this time. Consider three alternatives. The first is for all students taking mathematics in middle school. The last two are alternatives to algebra II for those students not planning on attending college or who are planning on non-quantitative majors in college—majors such as English, elementary education, history, political science, communications, journalism, music, art or philosophy.

- Introduce rates and percentages as presented in tables and graphs in middle school as a pre-algebra bridging course: a supplement to—or an application of—fractions.
- Introduce a Quantitative Literacy course as an alternative to algebra II. According to Gillman (2006), "there is consensus that the mathematical skills necessary to be quantitatively literate include elementary logic, the basic mathematics of financial interest, descriptive statistics, finite probability, an elementary understanding of change, the ability to model problems with linear and exponential models, estimations and approximation, and general problem solving." For more on such a course, see Gillman (2006) and Madison (2006).
- Introduce a Statistical Literacy course—evaluating statistical associations as evidence for causal connections—as an alternative to Algebra II. In addition to teaching students about rates, percentages, comparisons and standardization as devices for taking into account the influence of context, Statistical literacy could include a stronger focus on the influence of chance and include the influence of social construction —the choices made in defining groups or measures, in combining subgroups and in presenting statistical results in graphs, tables and in words. See Best (2001, 2002, 2004 and 2007) and Schield (2007a). For an overview of a Statistical Literacy course, see Schield (2004a, 2007b) and Isaacson (2005). This statistical literacy course could serve as a bridging course for those students wanting to take AP Statistics in high school.

8. Identify and teach topics that college students in non-quantitative majors need to master at the school level and which are currently not being taught there. Mastering percentages, rates and weighted averages allows students to take on more subtle mathematical and statistical topics that are commonly found in the everyday media, such as:

- *Simpson's paradox*: Suppose that a city hospital has a higher death rate among patients than does a rural hospital. But when patients are classified as being in either good or fair condition, patients in each condition have a lower death rate at the city than at the rural hospital (Schield 2006b).

- *Standardization*: Standardization takes into account the influence of a confounder using algebra or a graphical technique. Given the average family income for white and black families by type of family (single parent vs. married couple) and given the percentage of married couple families in each race, what percentage of the U.S. black-white family income gap is explained by differences in family structure? (Schield 2006b).
- *Cases attributable*: In the U.S. in 2003, the poverty rate was 25% in single-parent homes (5% in married-family homes). There are 4.5 million single-parent homes. How many of the single-parent families in poverty are attributable to their being a single-parent family? (Schield 2004b).
- *Bayes comparison*: Men are 94% of those in prison but 49% of the U.S. population so men are almost twice as prevalent among those in prison as among those in the general population. Using Bayes rule, we can conclude that men are almost twice as likely to go to prison as are those in the general population (Schield 2004b).

Conclusion

In preparing students for four-year colleges, school mathematics educators must justify their choice of topics and pedagogy for the 40% of college students who will graduate in non-quantitative majors. Satisfying the needs of this group is critical. These students are more likely to become journalists, policy advocates, lawyers, opinion makers and political leaders, thereby influencing local and national policies. College students in non-quantitative majors need quantitative literacy—even if they cannot (and need not) solve a quadratic equation or factor a cubic expression.

Whenever possible, school mathematics educators should look for ways to use context (the quantitative elements of everyday life) to drive the choice of quantitative topics rather than selecting mathematical topics and then looking for contexts in which it is used. Mathematics educators should focus more on those mathematical topics that are encountered most often in everyday contexts and that teachers in all majors can understand and will expect of their students. "Mathematics in context" should focus less on going from mathematics to context and focus more on going from context to mathematics.

In short, describing, comparing and standardizing percentages, rates and averages in context—in graphs, in tables and in ordinary English statements—should be an important element in the "mathematics in context" curriculum for both primary and secondary school students.

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