Statistical Literacy: The Lognormal Distribution

Milo Schield, Augsburg U.
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www.StatLit.org/
pdf/2018-Schield-ASA-Slides.pdf
XLS/Explore-LogNormal-Incomes-Excel2013.xlsx

Best-selling statistics books

80 million: *World Almanac (Since 1896)*

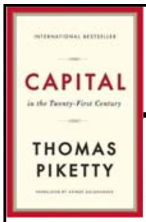
5 million: Economist: *World in Figures* (200K/yr; 25 years)

1.5 million: Piketty (2017): *Capital in the 21st Century*

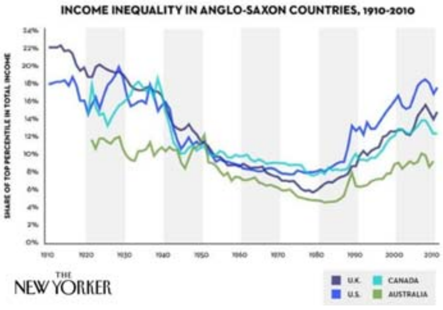
500,000: Murray & Hernstein (1994): *The Bell Curve*

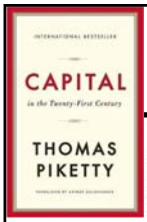
200,000: Hacker (1992): *"Two Nations: Black and White, Separate, Hostile, Unequal."*

https://www.washingtonpost.com/archive/lifestyle/1995/09/22/black-and-white-read-all-over-the-hot-books-that-make-the-melting-pot-boil/ee1de9b5-a172-4dfd-bb7a-1eb1d6cf9d77/



Capital in the 21st Century: Income by Country (Top 1%)





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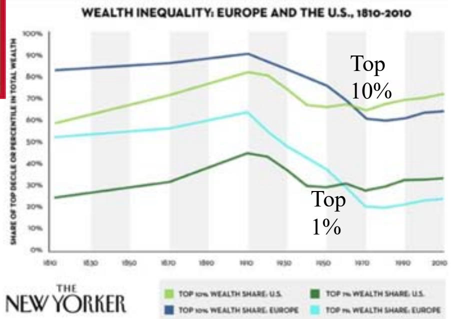
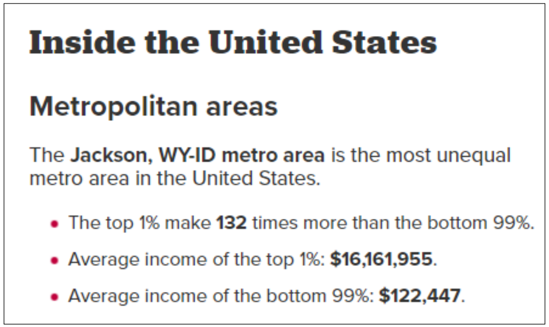


Top 10%

Top 1%

EPI (2018): US Income Inequality by Metro Area



Inside the United States

Metropolitan areas

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- The top 1% make **132** times more than the bottom 99%.
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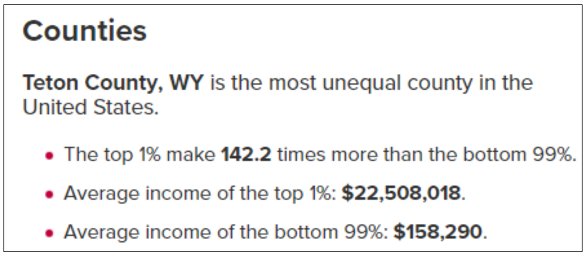

Counties

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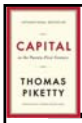

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Evaluate income share held by top 1% over time.

Data source: Tax data

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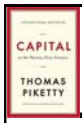

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"if β = 2, the average income of individuals with income above $100,000 is $200,000 and the average income of individuals with income above $1 million is $2 million."

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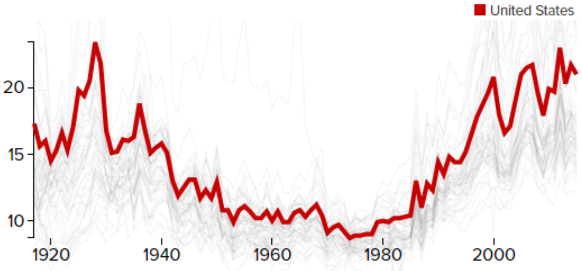
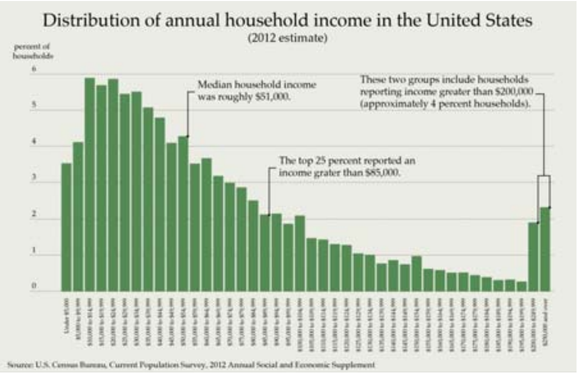
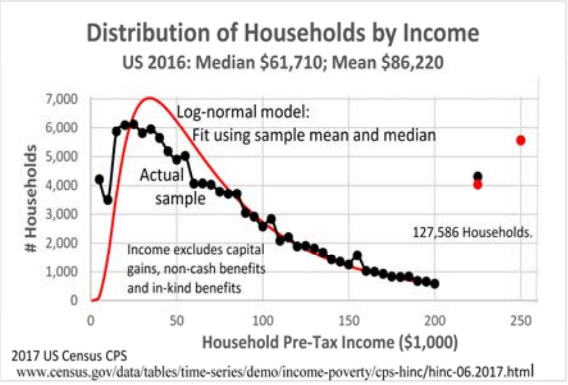



Distribution of annual household income in the United States
(2012 estimate)

Source: U.S. Census Bureau, Current Population Survey, 2012 Annual Social and Economic Supplement


Distribution of Households by Income
US 2016: Median $61,710; Mean $86,220

2017 US Census CPS
www.census.gov/data/tables/time-series/demo/income-poverty/cps-hinc/hinc-06.2017.html

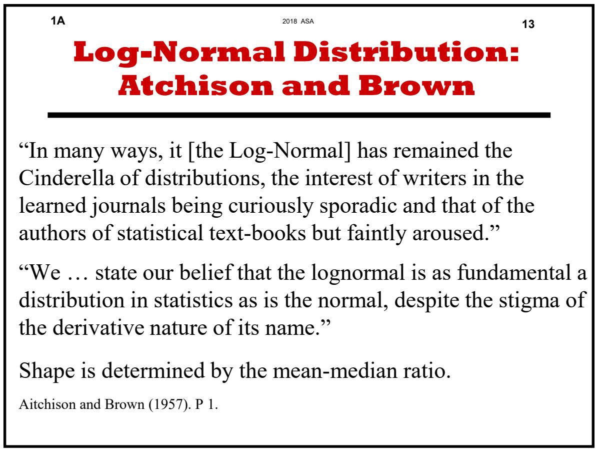
Log-Normal Distribution

Log-Normal shape is common. Examples:

- Incomes (bottom 97%), assets, size of cities
- Weight and blood pressure of humans (by gender)
- Stock and portfolio returns

Log-Normal is useful.

- Function is easier to work with than a histogram
- Understand what determines or explains shape
- calculate the share of total income held by the top X%
- calculate share of total income held by the 'above-average'
- explore effects of change in mean-median ratio.



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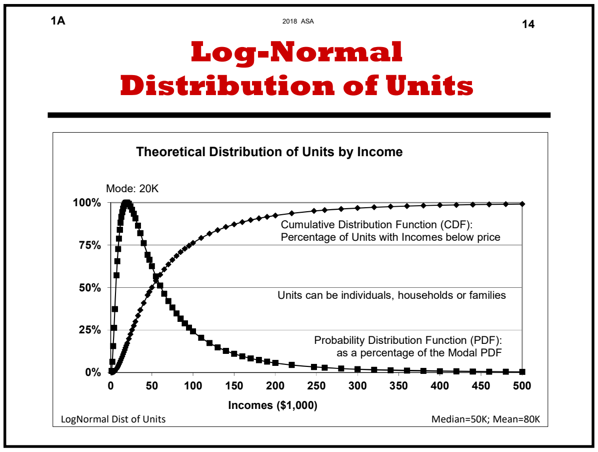
Log-Normal Distribution: Atchison and Brown

"In many ways, it [the Log-Normal] has remained the Cinderella of distributions, the interest of writers in the learned journals being curiously sporadic and that of the authors of statistical text-books but faintly aroused."

"We … state our belief that the lognormal is as fundamental a distribution in statistics as is the normal, despite the stigma of the derivative nature of its name."

Shape is determined by the mean-median ratio.

Aitchison and Brown (1957). P 1.



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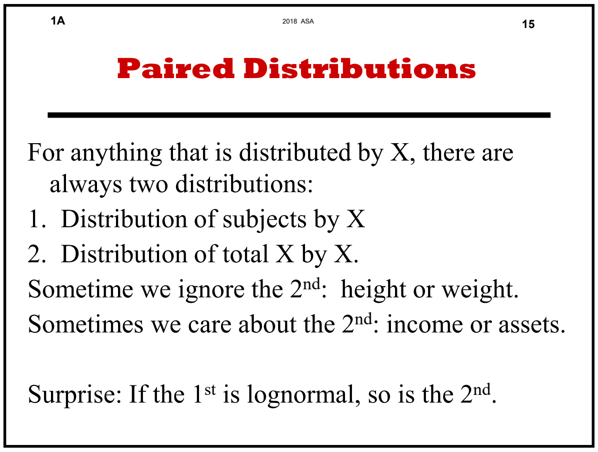
Log-Normal Distribution of Units

Theoretical Distribution of Units by Income

LogNormal Dist of Units Median=50K; Mean=80K



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Paired Distributions

For anything that is distributed by X, there are always two distributions:

1. Distribution of subjects by X
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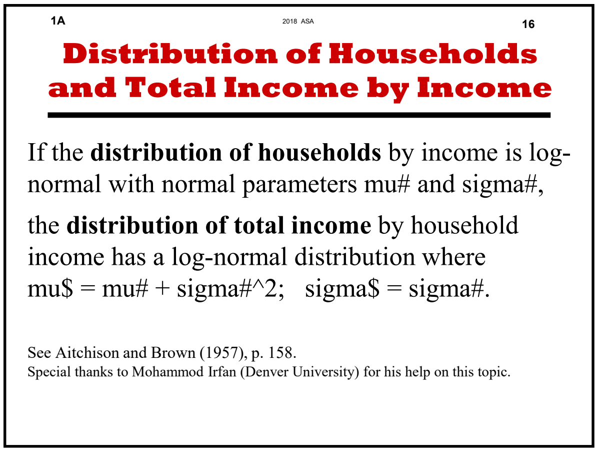
Sometime we ignore the 2nd: height or weight.

Sometimes we care about the 2nd: income or assets.

Surprise: If the 1st is lognormal, so is the 2nd.



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Distribution of Households and Total Income by Income

If the **distribution of households** by income is log-normal with normal parameters mu# and sigma#,

the **distribution of total income** by household income has a log-normal distribution where mu$ = mu# + sigma#^2; sigma$ = sigma#.

See Aitchison and Brown (1957), p. 158.
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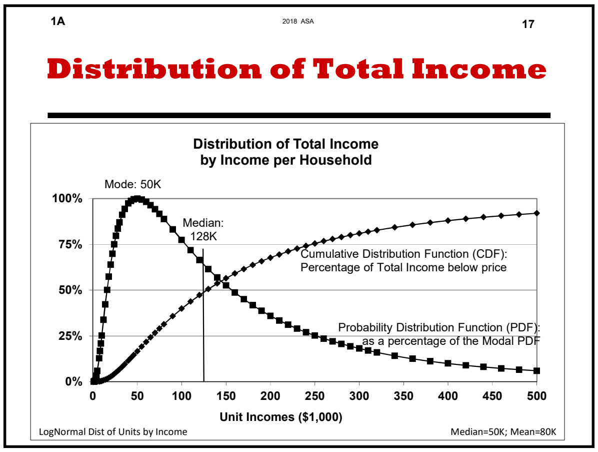

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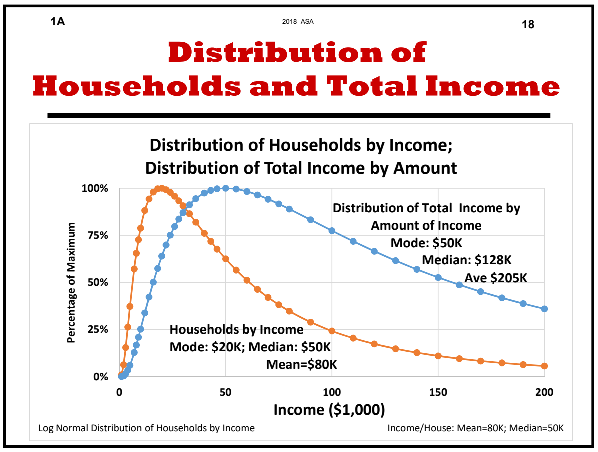
Distribution of Total Income

Distribution of Total Income by Income per Household

LogNormal Dist of Units by Income Median=50K; Mean=80K



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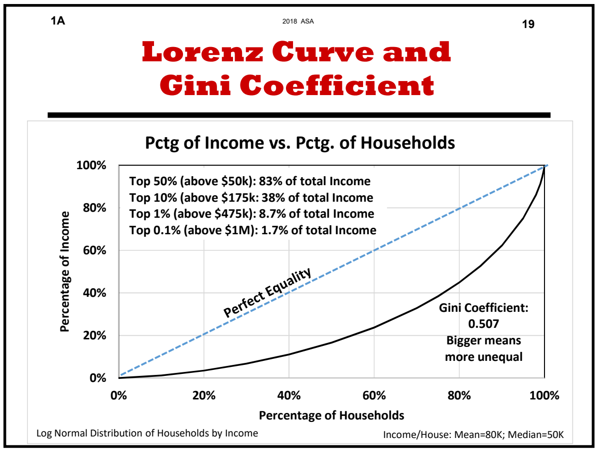
Distribution of Households and Total Income

Distribution of Households by Income; Distribution of Total Income by Amount

Log Normal Distribution of Households by Income Income/House: Mean=80K; Median=50K



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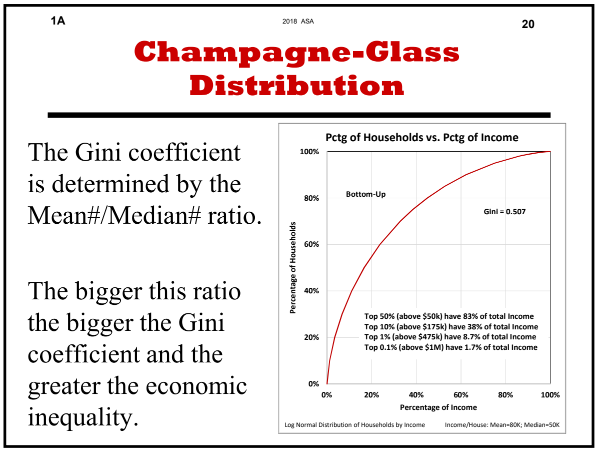
Lorenz Curve and Gini Coefficient



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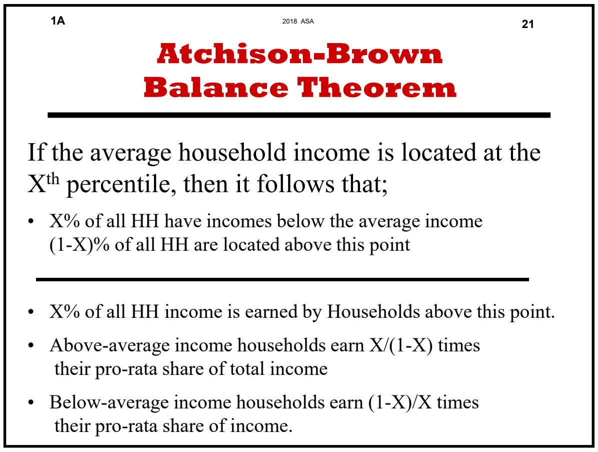
Champagne-Glass Distribution

The Gini coefficient is determined by the Mean#/Median# ratio.

The bigger this ratio the bigger the Gini coefficient and the greater the economic inequality.



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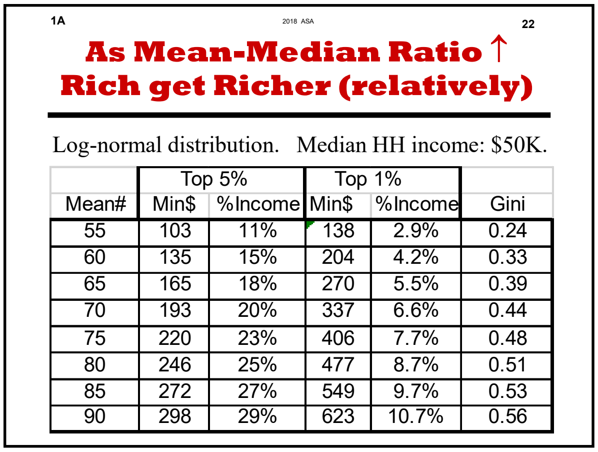
Atchison-Brown Balance Theorem

If the average household income is located at the Xth percentile, then it follows that;

- X% of all HH have incomes below the average income
  (1-X)% of all HH are located above this point

- X% of all HH income is earned by Households above this point.
- Above-average income households earn X/(1-X) times their pro-rata share of total income
- Below-average income households earn (1-X)/X times their pro-rata share of income.



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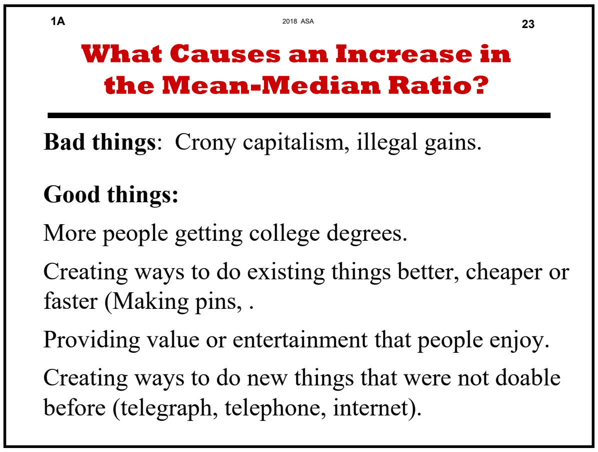
As Mean-Median Ratio ↑ Rich get Richer (relatively)

Log-normal distribution. Median HH income: $50K.

	Top 5%		Top 1%		
Mean#	Min$	%Income	Min$	%Income	Gini
55	103	11%	138	2.9%	0.24
60	135	15%	204	4.2%	0.33
65	165	18%	270	5.5%	0.39
70	193	20%	337	6.6%	0.44
75	220	23%	406	7.7%	0.48
80	246	25%	477	8.7%	0.51
85	272	27%	549	9.7%	0.53
90	298	29%	623	10.7%	0.56



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What Causes an Increase in the Mean-Median Ratio?

Bad things: Crony capitalism, illegal gains.

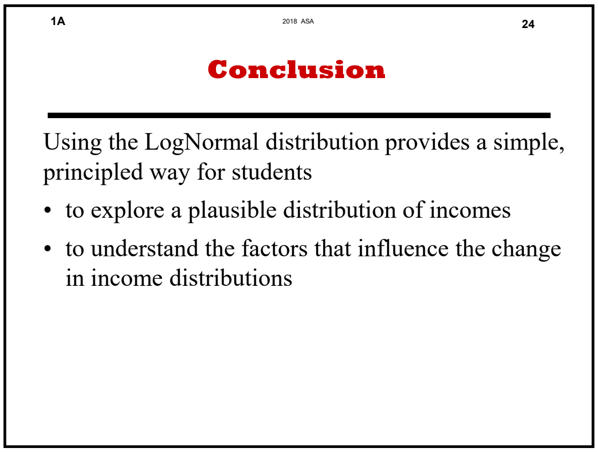
Good things:

More people getting college degrees.

Creating ways to do existing things better, cheaper or faster (Making pins, .

Providing value or entertainment that people enjoy.

Creating ways to do new things that were not doable before (telegraph, telephone, internet).



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Conclusion

Using the LogNormal distribution provides a simple, principled way for students

- to explore a plausible distribution of incomes
- to understand the factors that influence the change in income distributions

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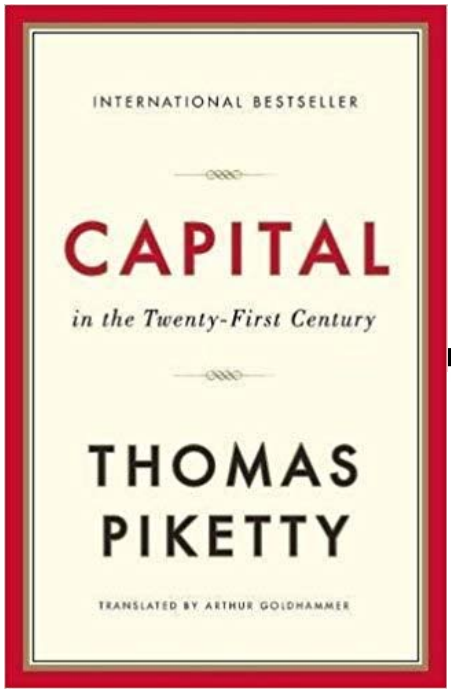


INTERNATIONAL BESTSELLER

CAPITAL

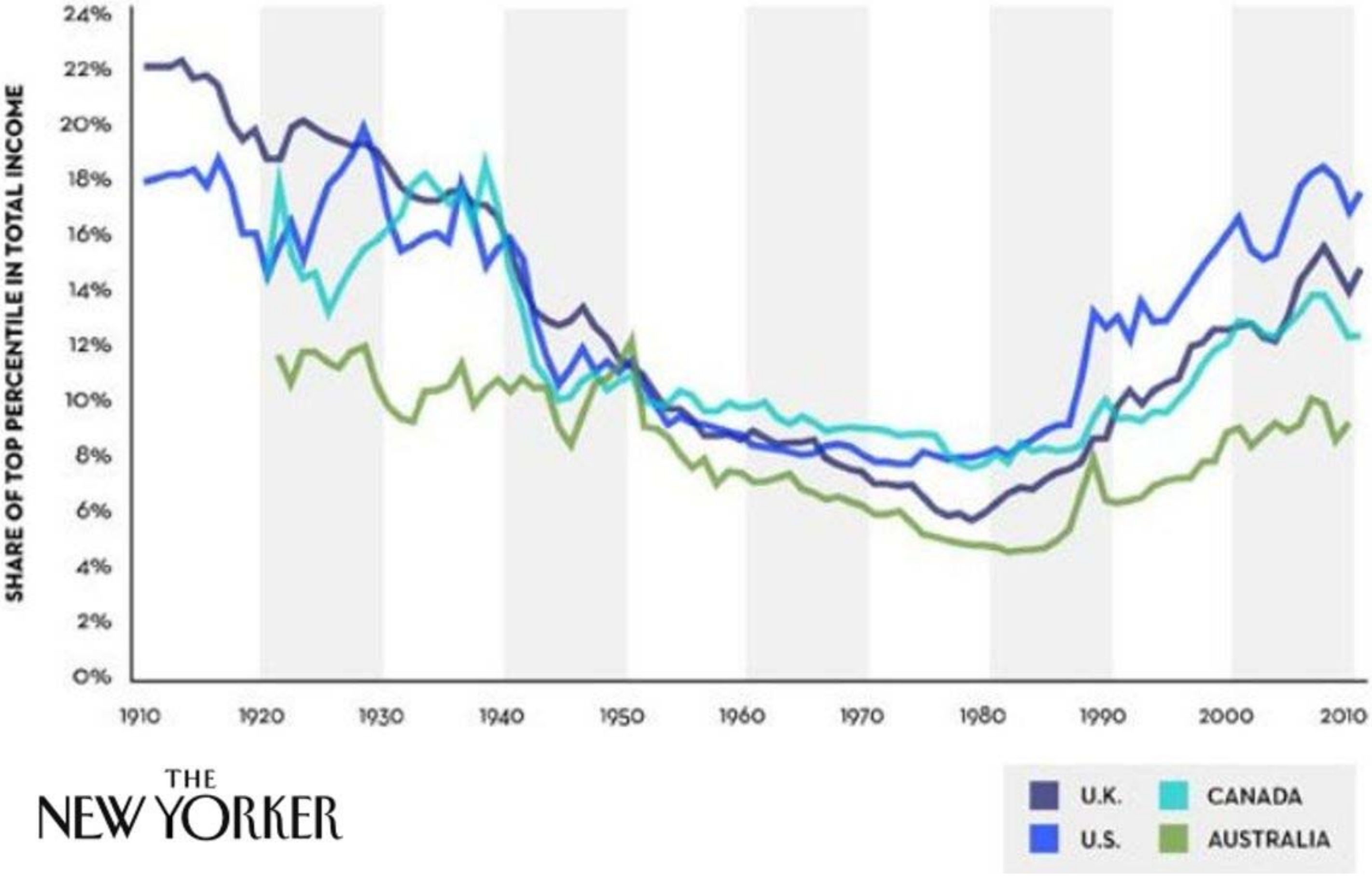
in the Twenty-First Century

THOMAS PIKETTY

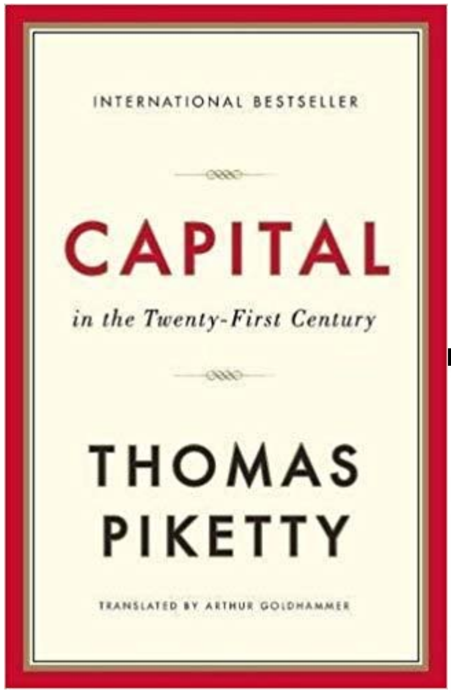
TRANSLATED BY ARTHUR GOLDHAMMER

Capital in the 21st Century: Income by Country (Top 1%)

INCOME INEQUALITY IN ANGLO-SAXON COUNTRIES, 1910-2010



THE NEW YORKER



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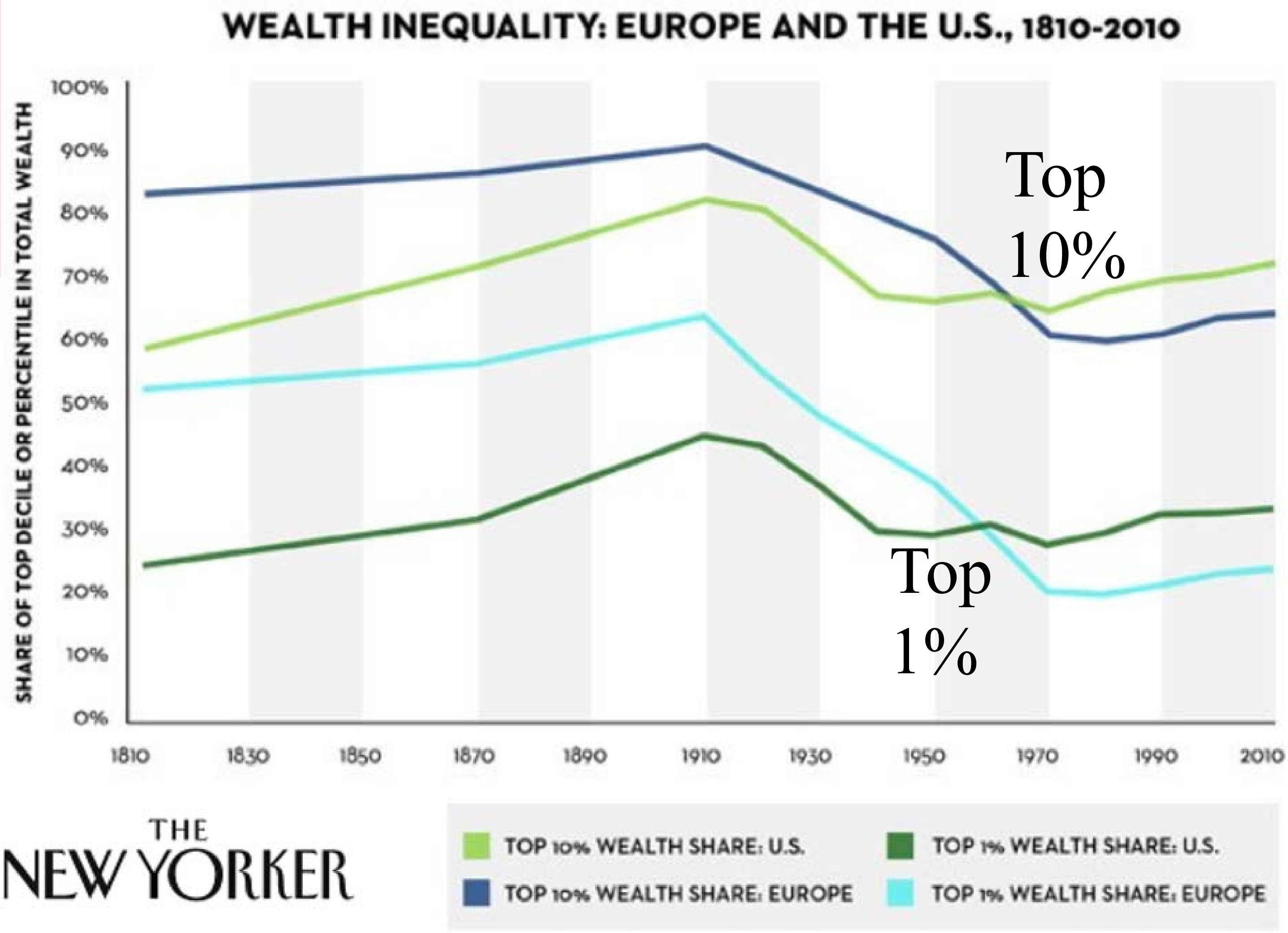
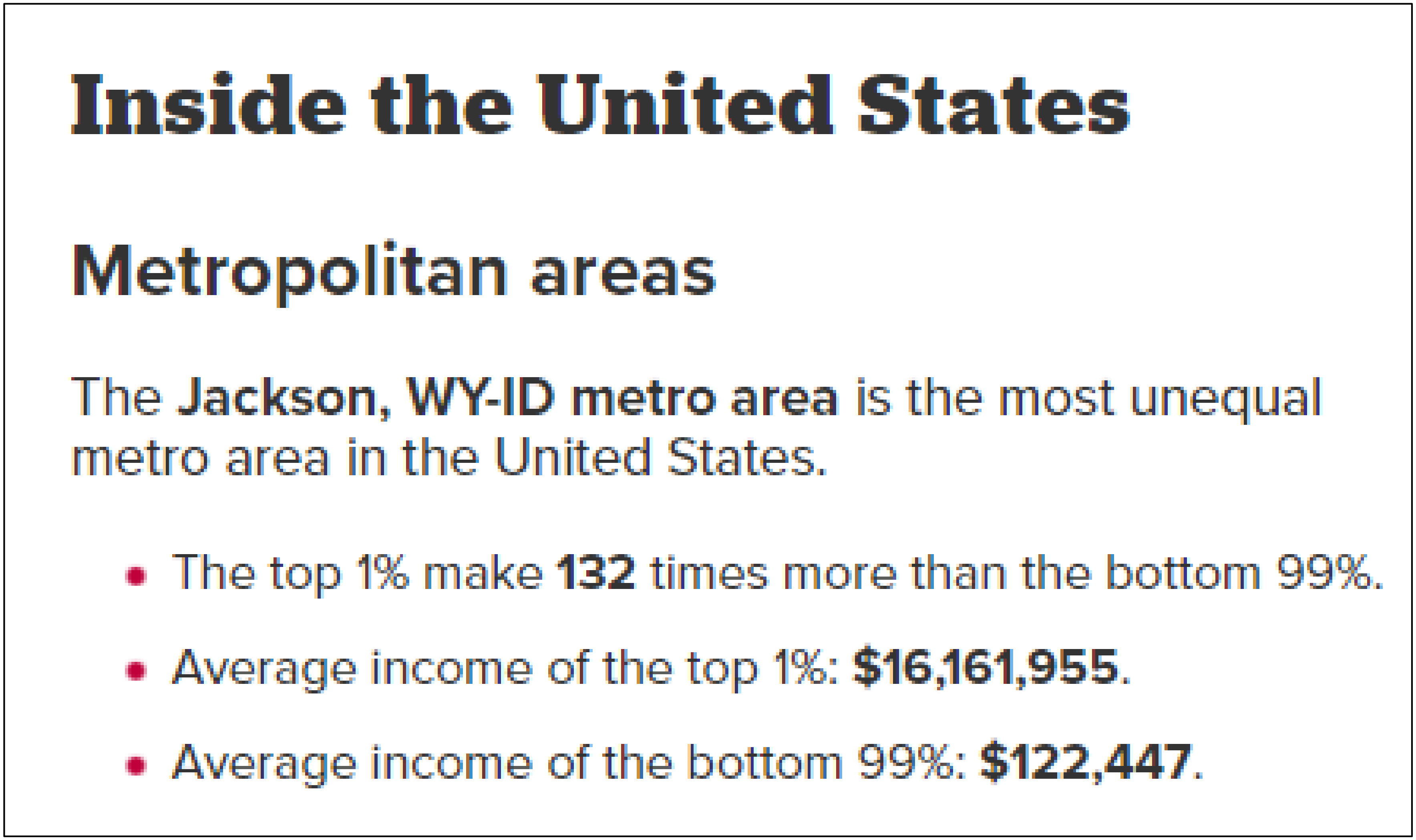


Top 10%

Top 1%

THE NEW YORKER

EPI (2018): US Income Inequality by Metro Area



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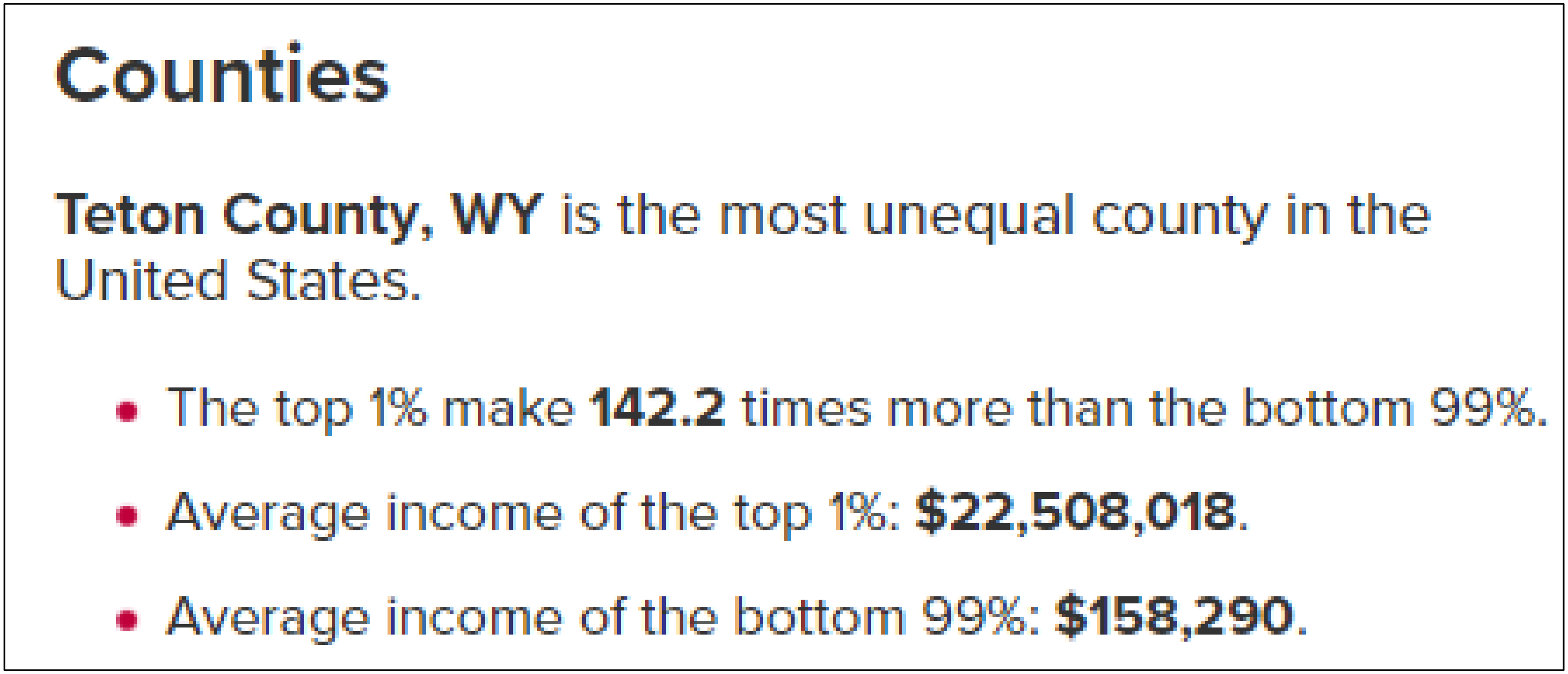

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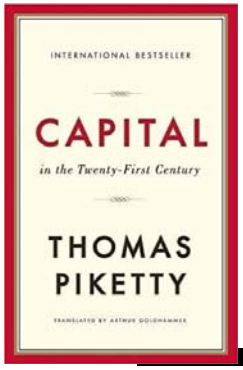

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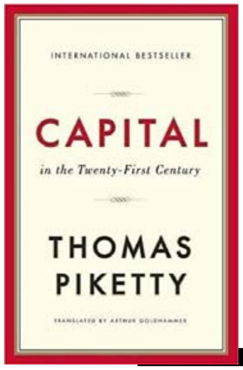

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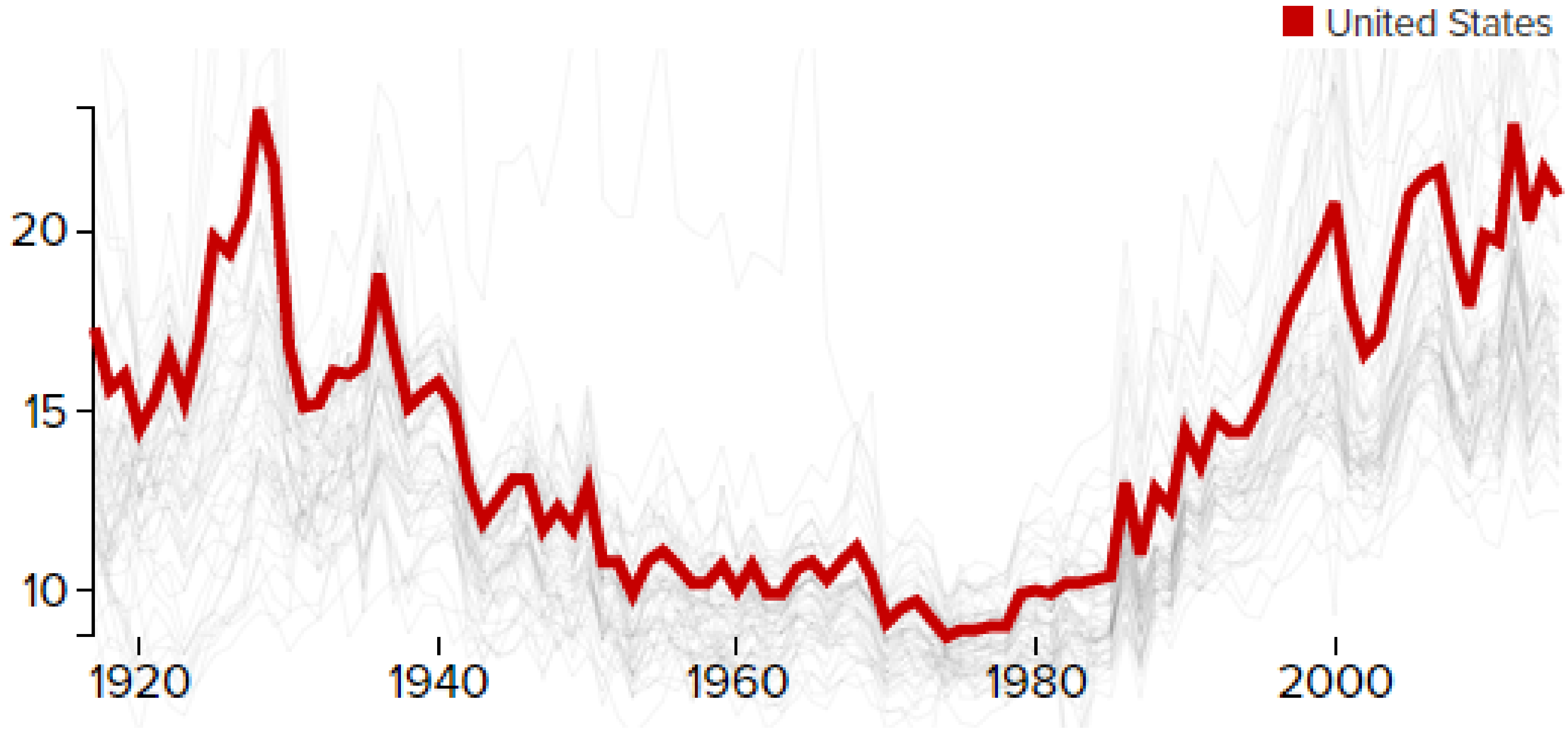
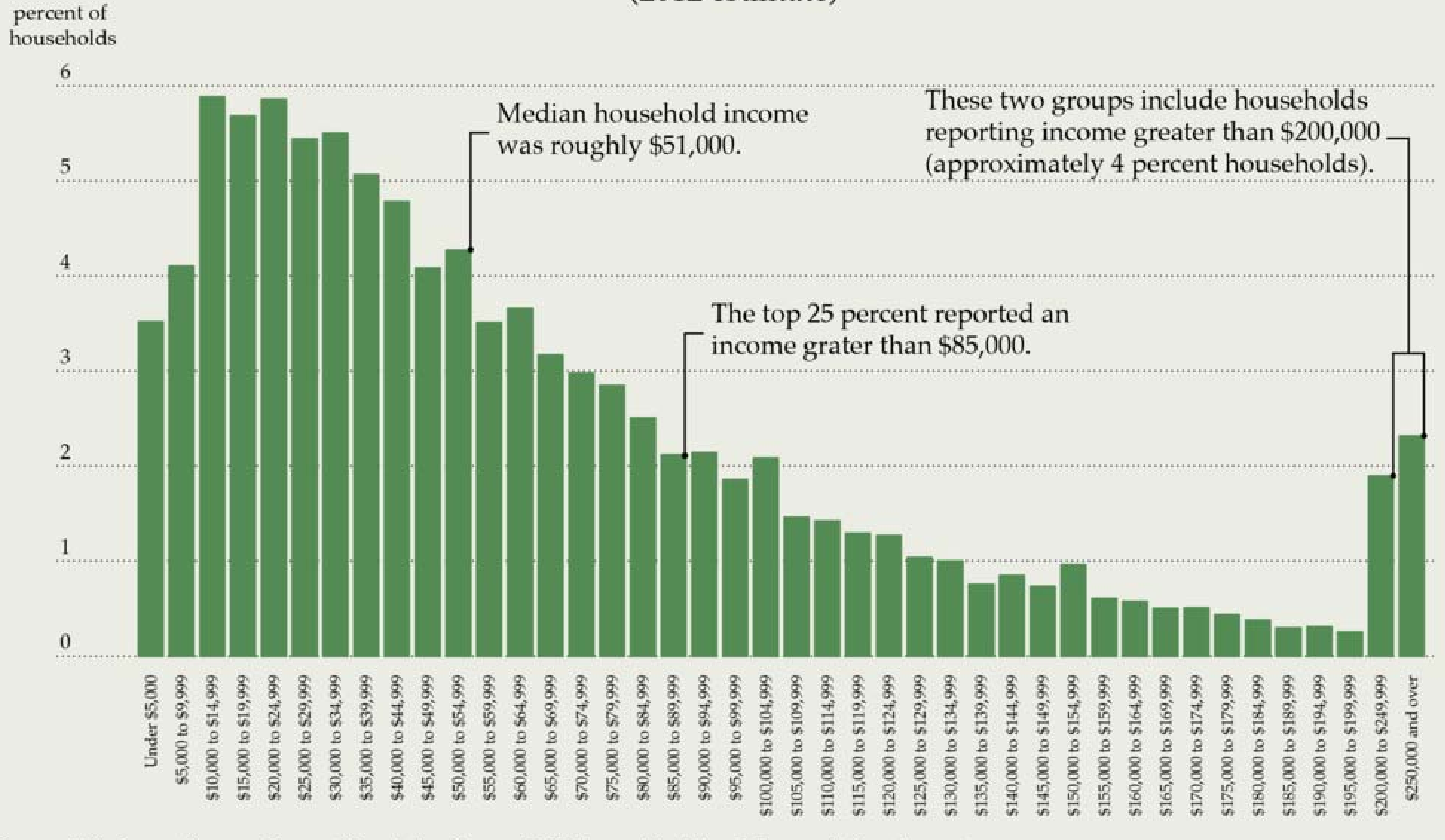


Distribution of annual household income in the United States

(2012 estimate)



Median household income was roughly $51,000.

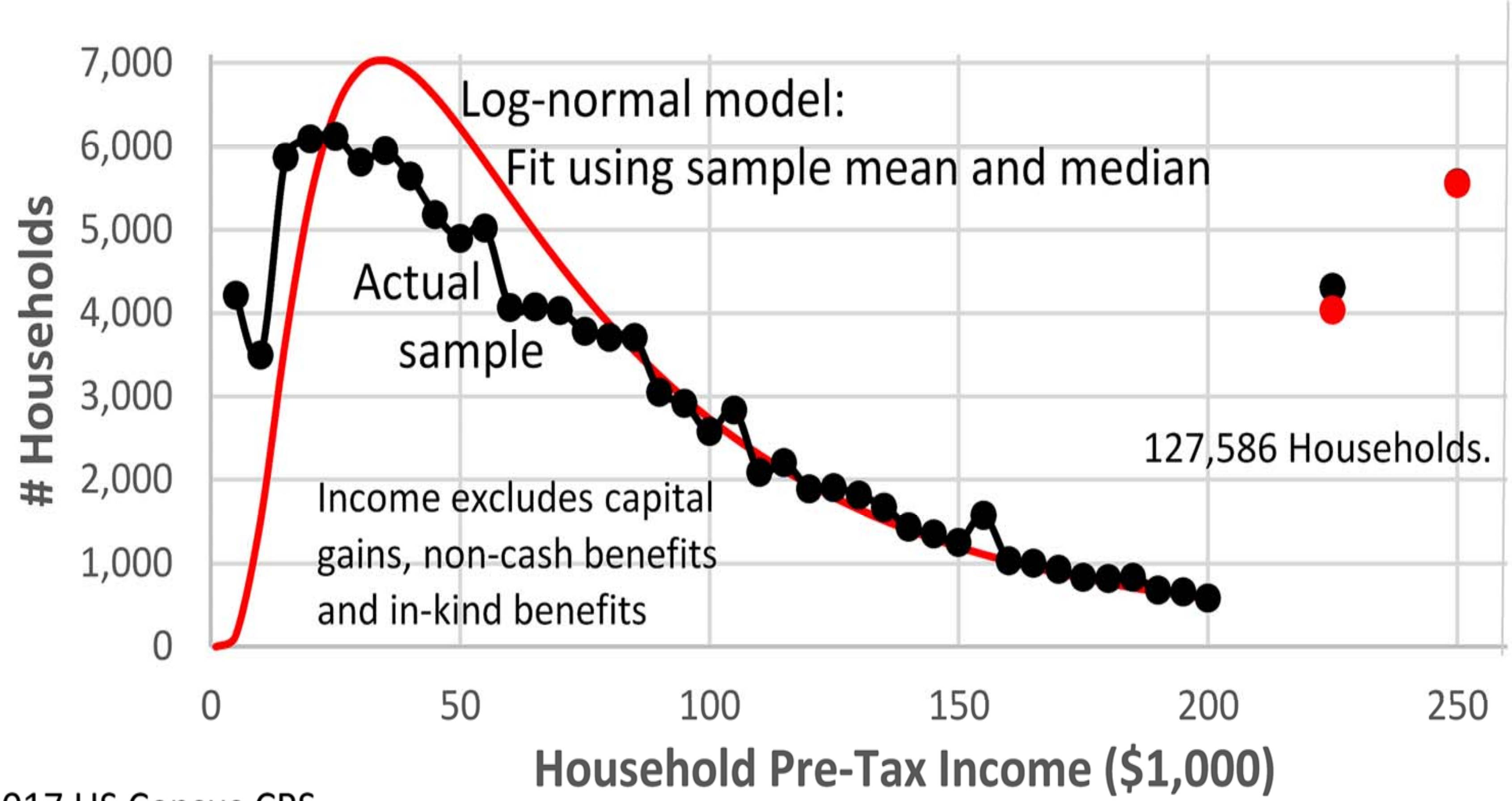
These two groups include households reporting income greater than $200,000 (approximately 4 percent households).

The top 25 percent reported an income grater than $85,000.

Source: U.S. Census Bureau, Current Population Survey, 2012 Annual Social and Economic Supplement

Distribution of Households by Income

US 2016: Median $61,710; Mean $86,220



Log-normal model:
Fit using sample mean and median

Actual
sample

127,586 Households.

Income excludes capital
gains, non-cash benefits
and in-kind benefits

2017 US Census CPS
www.census.gov/data/tables/time-series/demo/income-poverty/cps-hinc/hinc-06.2017.html

Log-Normal Distribution

Log-Normal shape is common. Examples:

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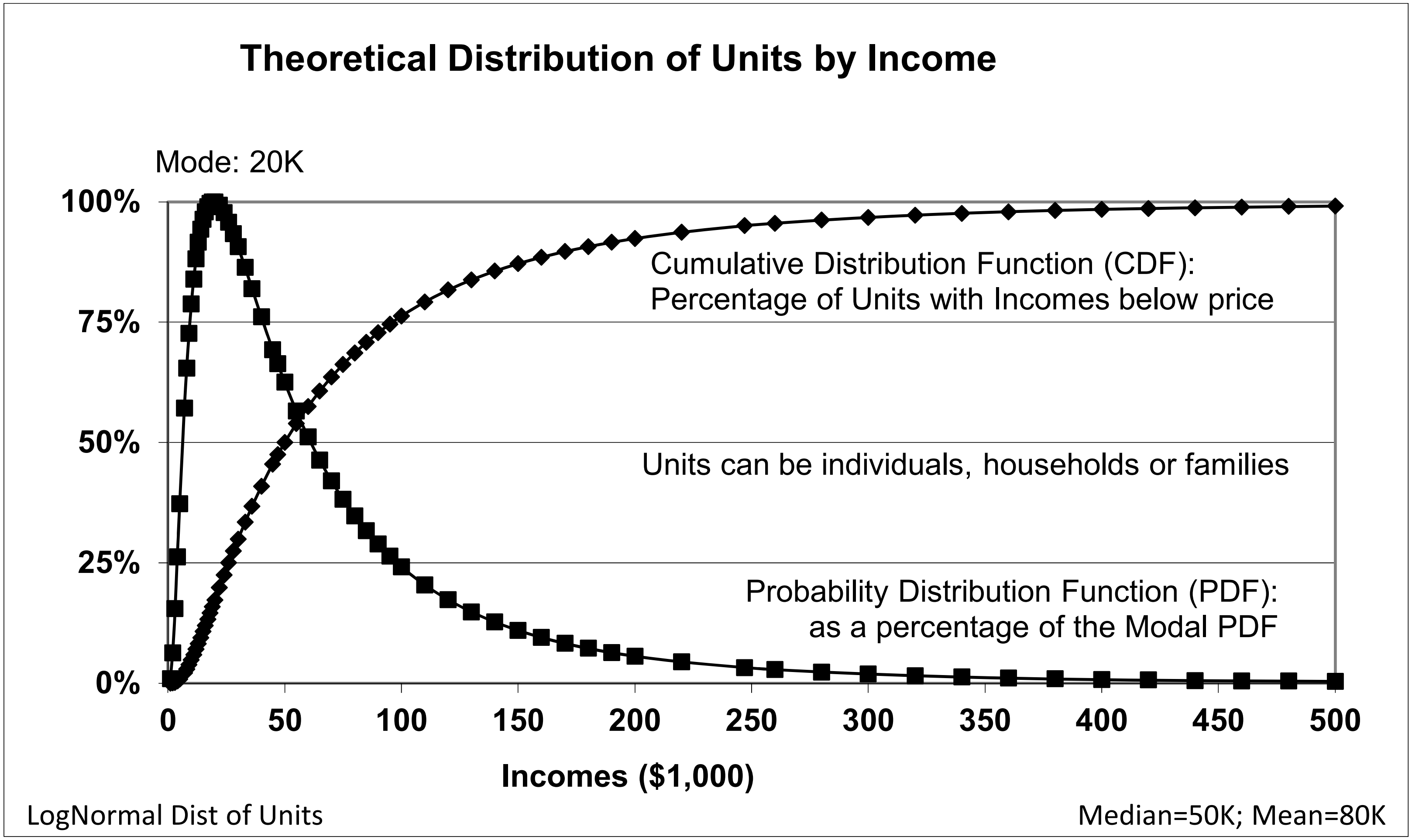
Log-Normal Distribution of Units



Paired Distributions

For anything that is distributed by X, there are always two distributions:

1. Distribution of subjects by X
2. Distribution of total X by X.

Sometime we ignore the 2nd: height or weight.

Sometimes we care about the 2nd: income or assets.

Surprise: If the 1st is lognormal, so is the 2nd.

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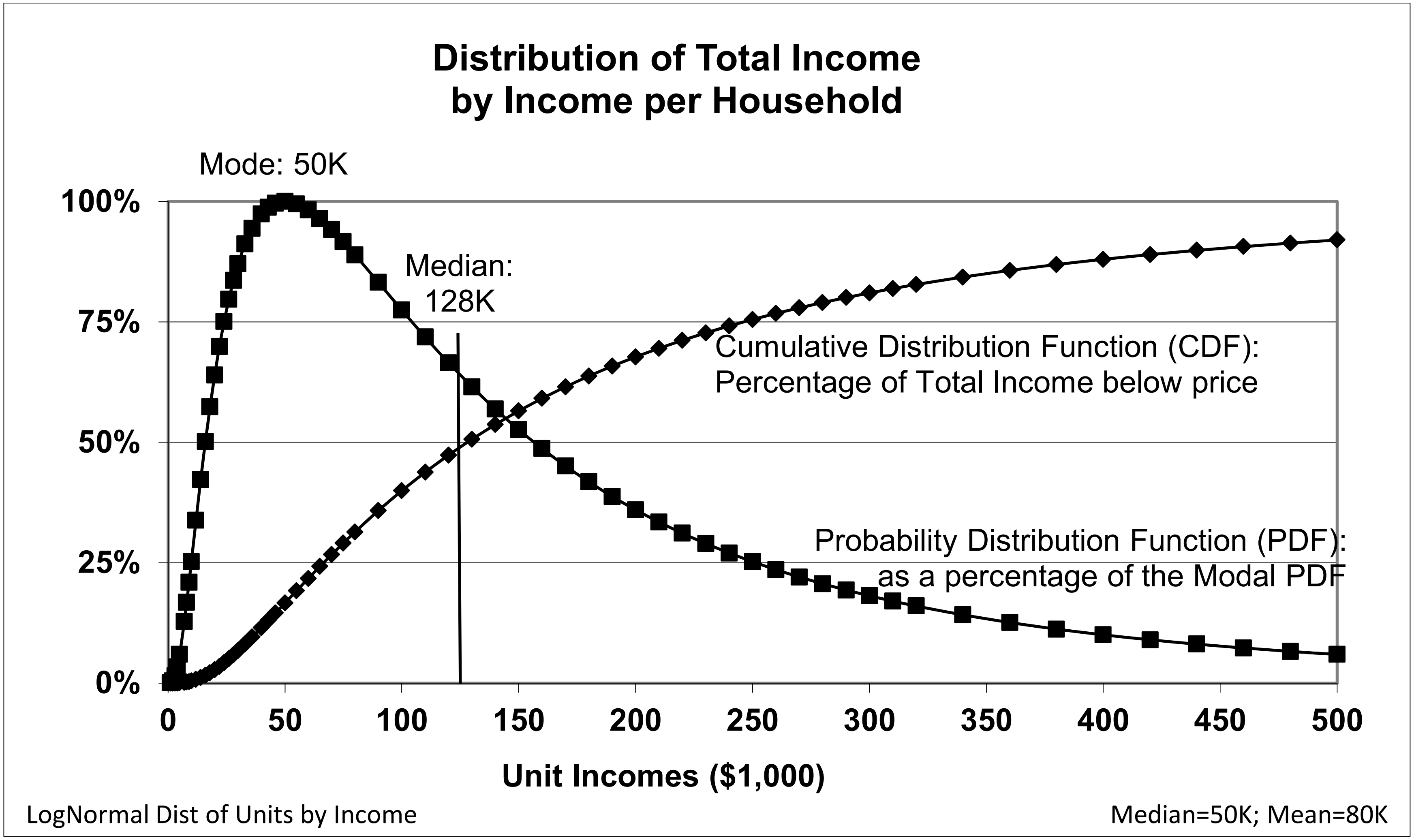
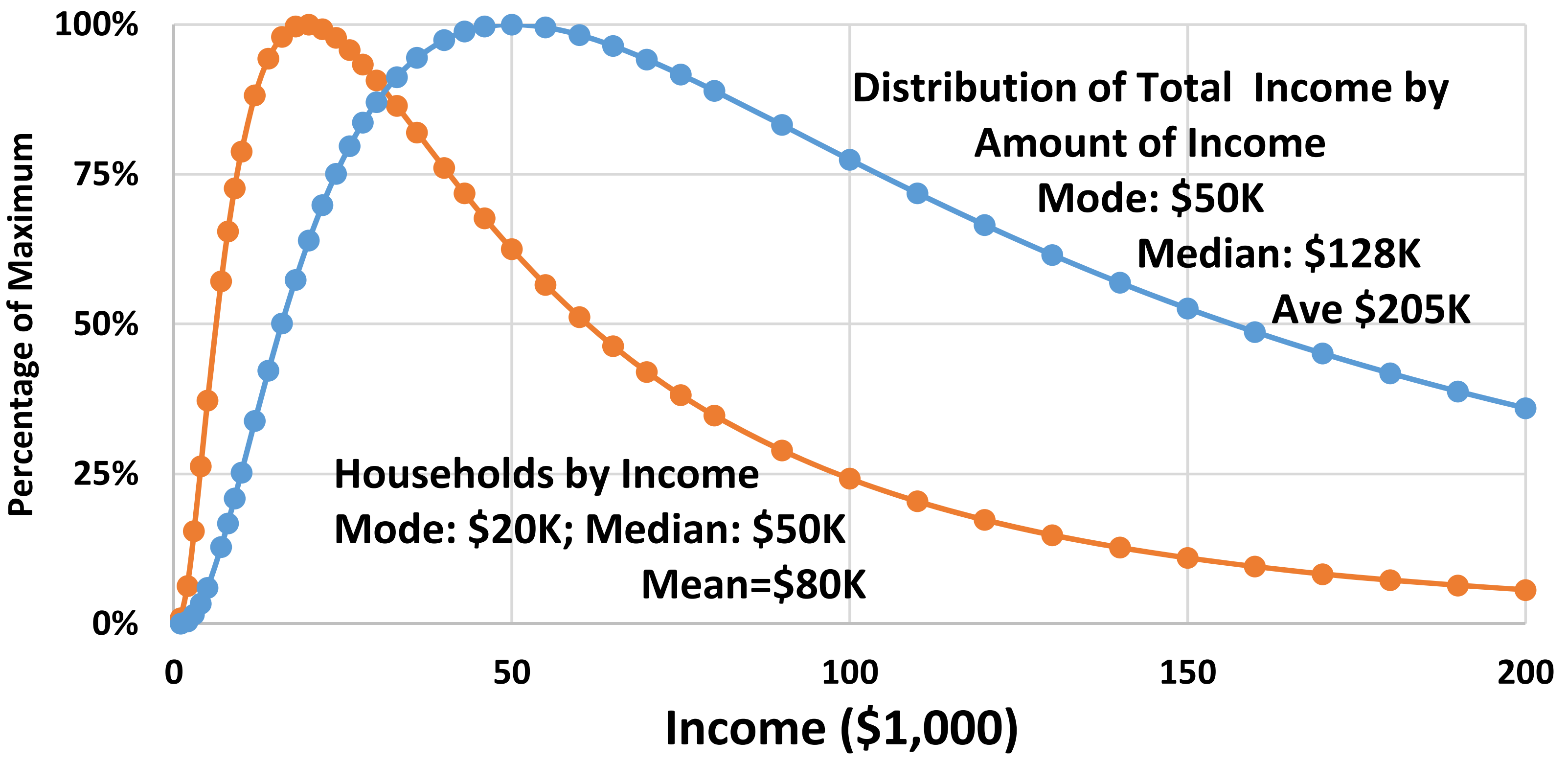


LogNormal Dist of Units by Income

Median=50K; Mean=80K

Distribution of Households and Total Income

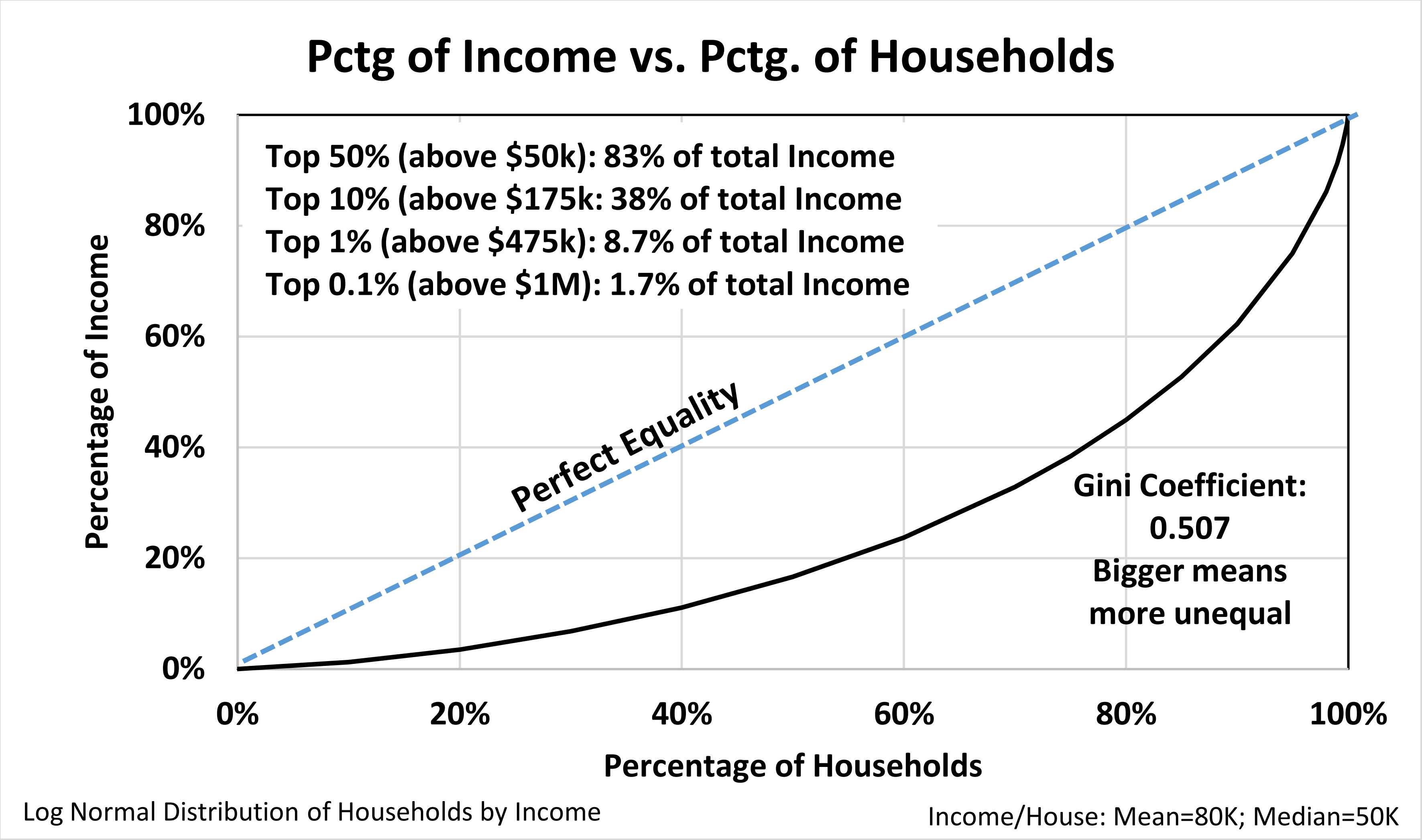
Distribution of Households by Income;
Distribution of Total Income by Amount



Log Normal Distribution of Households by Income

Income/House: Mean=80K; Median=50K

Lorenz Curve and Gini Coefficient



Top 50% (above $50k): 83% of total Income
Top 10% (above $175k: 38% of total Income
Top 1% (above $475k): 8.7% of total Income
Top 0.1% (above $1M): 1.7% of total Income

Gini Coefficient:
0.507
Bigger means
more unequal

Log Normal Distribution of Households by Income

Income/House: Mean=80K; Median=50K

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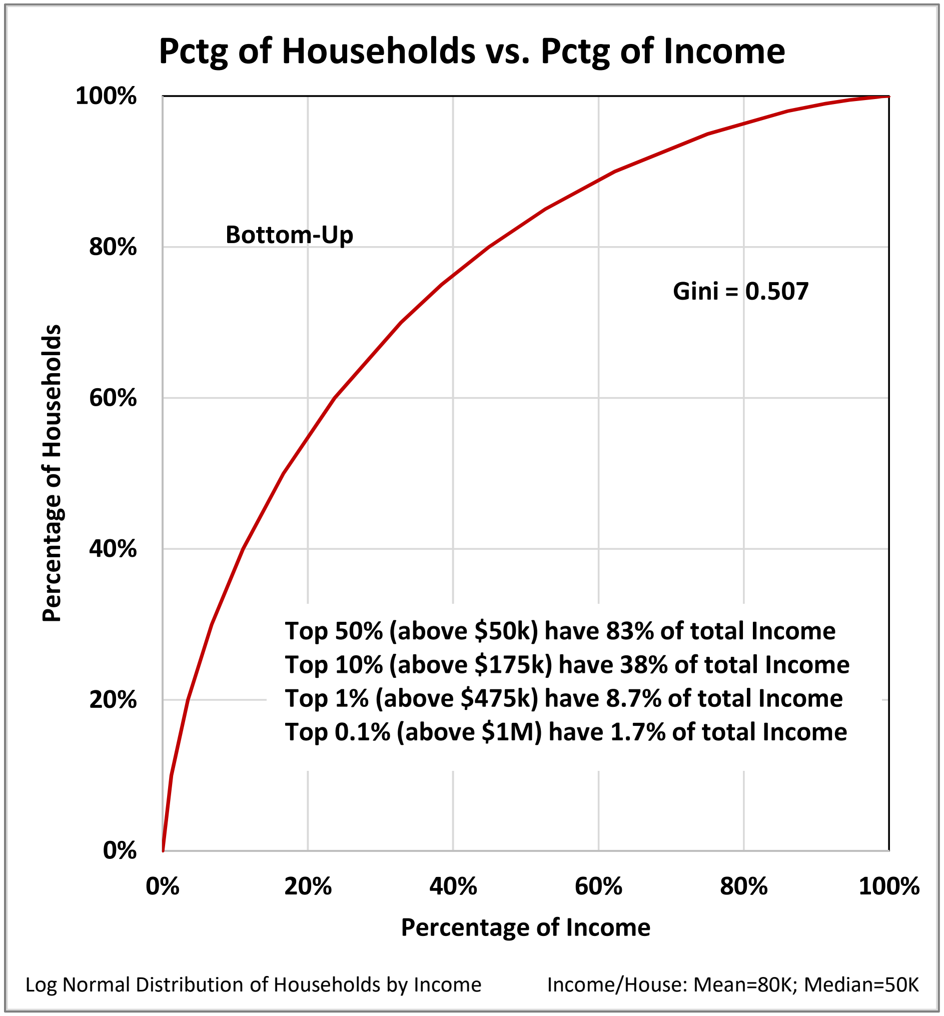

Pctg of Households vs. Pctg of Income

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Log Normal Distribution of Households by Income Income/House: Mean=80K; Median=50K

Atchison-Brown Balance Theorem

If the average household income is located at the Xth percentile, then it follows that;

- X% of all HH have incomes below the average income (1-X)% of all HH are located above this point

- X% of all HH income is earned by Households above this point.
- Above-average income households earn X/(1-X) times their pro-rata share of total income
- Below-average income households earn (1-X)/X times their pro-rata share of income.

As Mean-Median Ratio ↑ Rich get Richer (relatively)

Log-normal distribution. Median HH income: $50K.

	Top 5%		Top 1%		
Mean#	Min$	%Income	Min$	%Income	Gini
55	103	11%	138	2.9%	0.24
60	135	15%	204	4.2%	0.33
65	165	18%	270	5.5%	0.39
70	193	20%	337	6.6%	0.44
75	220	23%	406	7.7%	0.48
80	246	25%	477	8.7%	0.51
85	272	27%	549	9.7%	0.53
90	298	29%	623	10.7%	0.56

What Causes an Increase in the Mean-Median Ratio?

Bad things: Crony capitalism, illegal gains.

Good things:

More people getting college degrees.

Creating ways to do existing things better, cheaper or faster (Making pins, .

Providing value or entertainment that people enjoy.

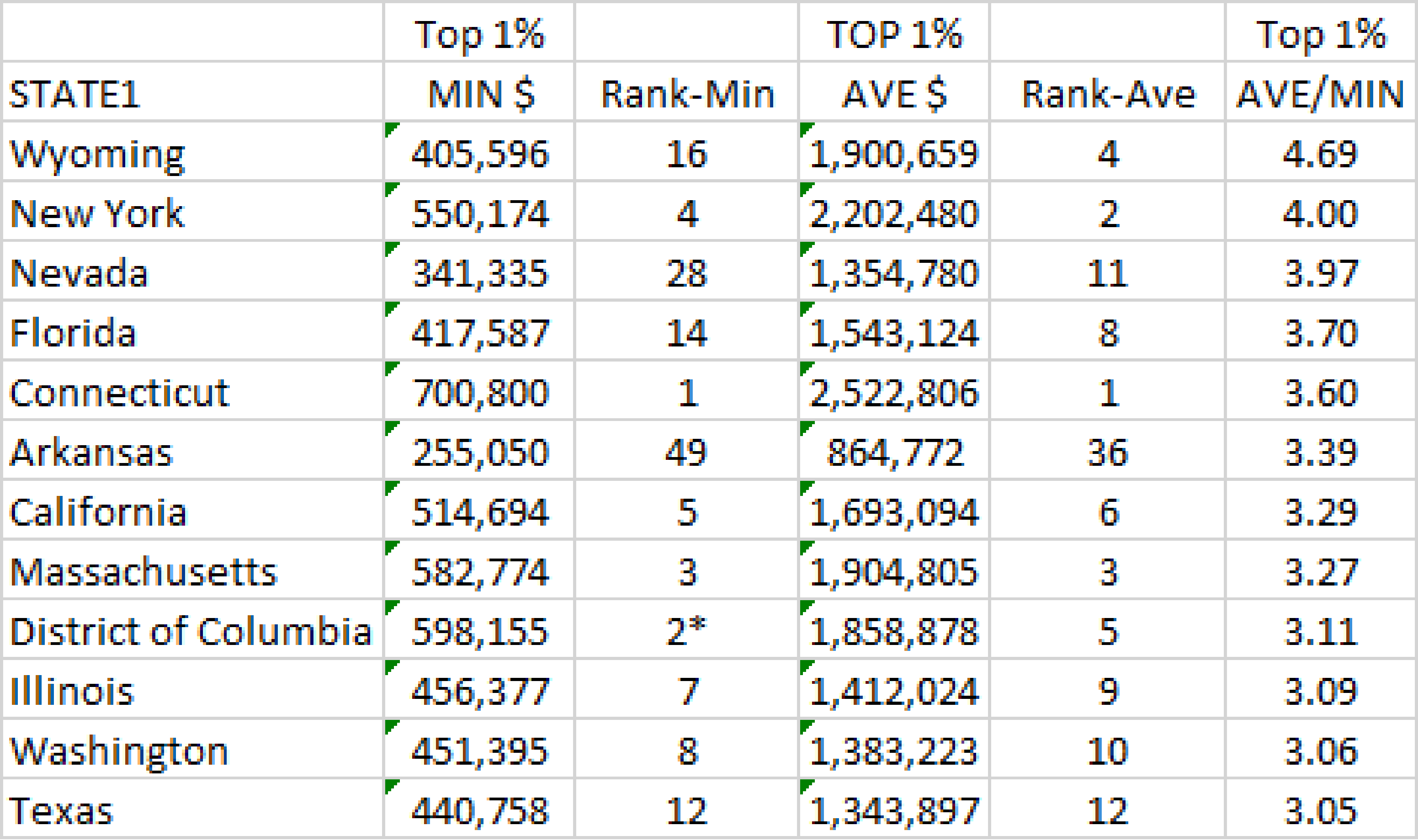
Creating ways to do new things that were not doable before (telegraph, telephone, internet).

Conclusion

Using the LogNormal distribution provides a simple, principled way for students

- to explore a plausible distribution of incomes
- to understand the factors that influence the change in income distributions

EPI (2018): US Income Inequality by State



	Top 1%		TOP 1%		Top 1%
STATE1	MIN $	Rank-Min	AVE $	Rank-Ave	AVE/MIN
Wyoming	405,596	16	1,900,659	4	4.69
New York	550,174	4	2,202,480	2	4.00
Nevada	341,335	28	1,354,780	11	3.97
Florida	417,587	14	1,543,124	8	3.70
Connecticut	700,800	1	2,522,806	1	3.60
Arkansas	255,050	49	864,772	36	3.39
California	514,694	5	1,693,094	6	3.29
Massachusetts	582,774	3	1,904,805	3	3.27
District of Columbia	598,155	2*	1,858,878	5	3.11
Illinois	456,377	7	1,412,024	9	3.09
Washington	451,395	8	1,383,223	10	3.06
Texas	440,758	12	1,343,897	12	3.05

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