Reflections on Mathematical Patterns, Relationships, and Functions

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Learning Goal: Students will represent and analyze mathematical patterns, relationships, and functions to model and solve problems.

With this learning goal in mind, Minnesota students will have the opportunity to pursue the following instructional components:

- Recognize, describe, and generalize patterns and build mathematical models to make predictions.
- Analyze the interaction between quantities and/or variables to model patterns of change.
- Use algebraic concepts and processes to represent and solve problems that involve variable quantities.

As biology is the science of life and physics the science of energy and matter, so mathematics is the science of patterns (Steen, 1988; Devlin, 1994). We live in an environment steeped in patterns--patterns of numbers and space, of science and art, of computation and imagination. Patterns permeate the learning of mathematics, beginning when children learn the rhythm of counting and continuing through times-tables all the way to fractals and binomial coefficients. Patterns also explain the extraordinary utility of mathematics, since understanding nature and predicting behavior is largely a matter of identifying appropriate patterns.

Patterns are often expressed through relationships and functions. Tables, charts, formulas, and graphs are among the many devices used to represent patterns and relationships (e.g., between Centigrade and Fahrenheit, miles and kilometers, speed and time, principle and interest). Relationships establish connections and document dependencies; they reveal correlations or confirm independence. Some relationships are direct, others inverse; some show steady trends, others oscillate; some are smooth, others discontinuous. Most can be represented in multiple forms--graphically, verbally, numerically, or analytically. Indeed, the patterns of mathematics are often revealed most effectively through the interplay of these multiple representations.

> "What humans do with the language of mathematics is describe patterns."
> (Steen, 1990, p. 8.)

The Rosetta Stone of relationships is the language of algebra. Most people think of algebra as a system of arithmetic with letters that is helpful for solving equations. Although today algebra is viewed as one of the most useful parts of mathematics, historically it was taught not to solve problems but to train minds (Katz, 1995). It was the logical discipline of doing algebra, not the solutions of its equations, that made it an essential part of classical education. Today it is a battlefield for educational reform: What kind of algebra is essential? When should it be taught? How should it be taught? Who should study it? Why is it important? Endless questions have led to endless debates (Lacampagne, 1995). But at the center of every debate is algebra, the language of mathematics (Artin, 1995).

"In algebra we are not just studying the relationships of number to one another, but we are also affirming the relationships of numbers to ourselves. . . . The notion that we can go from unknowns to knowns by identifying relationships, that we can find out what *we* want to know, is a very powerful idea." (Malcom, 1997, p. 32.)

In earlier periods, parents and educators conducted school mathematics within a well-established tradition of separate tracks: algebra for the elite, arithmetic for the masses. While this system may have been justifiable at a time when over half the labor force worked in unskilled jobs, it is being challenged today on several grounds. First, many new jobs require at least some familiarity with algebra. For example, anyone who uses computer spreadsheets needs to understand algebraic notation. Second, most of our international competitors teach algebraic thinking over many years of the upper elementary grades, thereby making it a universal part of school education. Third, in today's technological society, algebra has become the symbol of access--what Robert Moses calls "the new civil right" (Moses, 1995). Algebra is both the key to advanced mathematics and the passport to economic security.

In the United States, algebra has come to be represented by the single course called Algebra I that forms the transition between elementary and secondary school. Most U.S. students take this course in ninth grade, although some take it earlier and others later (or not at all). Algebra I is defined by a content well known to parents from their own school experiences: algebraic formulas and expressions, factoring, linear equations, slopes, graphs, and word problems. This course introduces students to the abstractions of advanced mathematics and covers skills widely used in subsequent courses both in high school and college (Usiskin, 1995). Subsequent high school courses provide additional topics of increasing abstraction--quadratic equations, conic sections, trigonometric and transcendental functions.

"Algebra for all is the right goal at the right time. We just need to get the right algebra." (Donald L. Chambers, 1994, p. 30.)

Despite its continued popularity, this approach to algebra is an historical anachronism. In its present form, first-year algebra is not appropriate for today's needs (NCTM, 1994). To meet the mathematical needs of students who will live and work in the next century, the *Minnesota Frameworks* focuses instead on the more general theme of patterns and relationships. This theme encompasses traditional school algebra, but much more as well:

"Because algebra is very much like a language, ... it is better learned earlier and harder to learn when one is older." (Alan Schoenfeld, 1995, p. 12.)

- Algebraic thinking develops best from **continuous exposure** to patterns and relationships throughout school, beginning in kindergarten. Mastery of mathematical language, as of English or Spanish, requires years of experience. Much of algebra is an extension of arithmetic into the more abstract realm of quantities represented by letters. Although algebra thus follows arithmetic in a logical sense, for effective learning, algebraic thinking must be nurtured in parallel with arithmetic understanding. The *Minnesota Frameworks'* emphasis on patterns and relationships at all grade levels provides optimal conditions for the growth of algebraic thinking in all children.

- Patterns and relationships arise in **all parts of mathematics**--in numbers and chance, in geometry and data--and not just in the realm of formulas and functions. By liberating the concepts of functions and relations from the narrow confines of Algebra I, the *Minnesota Frameworks* encourages teachers to introduce students to the ligaments of mathematics, the connective tissue that holds the many parts together.

> "Patterns can be either real or imagined, visual or mental, static or dynamic, qualitative or quantitative. ... They can arise from the world around us ... or from the inner workings of the human mind."
> (Devlin, 1994, p. 3.)

- Most patterns are noticed first not in mathematics but in **the external world**--in nature, art, business, science, engineering, agriculture, and architecture. The broad scope of this Learning Goal extends into all these arenas, inviting teachers to collaborate with colleagues in other disciplines to help students see patterns wherever they arise. Experience with the interplay of real-world patterns and their mathematical representations lays the foundation for algebraic thinking.

> "A student's mathematics education is simply not complete if that student has not experienced the usefulness of mathematics in the larger world." (Pollak 1997, p. 105.)

- The mathematics people use most is **concrete mathematics**--elementary tools combined to solve complex problems. In ordinary life, simple tools such as arithmetic, ratios, and spreadsheets are far more common than are logarithmic or trigonometric functions. However, ordinary problems are nonetheless often subtle and complex: weighing alternative mortgage options, understanding the implications of a change in the tax code, designing a deck of maximum area with minimum cost. Broader experience with problems of this type helps provide a firm foundation for the abstractions encountered in advanced high school and collegiate mathematics.

> "Concrete mathematics is rich in data, interspersed with conjecture, dependent on technology, and tied to useful applications. ... It is specific but not narrow, focused but not prescribed. It is found embedded in rich, authentic examples that stimulate students to think mathematically."
> (Forman & Steen, p. 227-8.)

- The ubiquitous spread of technology, especially **calculators and spreadsheets**, has changed significantly the relative importance of different topics in the traditional curriculum. On the one hand, the presence of these calculating devices on every desktop means that paper and pencil methods no longer represent "best practice" in mathematics. On the other hand, since algebra is the language of spreadsheets, and since calculator results always need testing against *a priori* estimates, the increasing reliance on machine calculations renders even more important a working understanding of mathematical relations and patterns.

> "Technology makes mathematics realistic."
> (MSEB, 1990, p. 21.)

- The increasing capabilities of computer graphics have enhanced the importance of **visualization** as a feature of modern mathematics. From hand-held calculators to portable computers, mathematics these days is as often represented by graphs and images as

> "For centuries the mind has dominated the eye in the hierarchy of mathematical practice; today the balance is being restored as mathematicians find new ways to see patterns, both with the eye and with the mind." (Steen, 1990, p. 2.)

by symbolic or numerical forms. Visual representation is often the best way to "see" a pattern, since it takes advantage of the visual intelligence of our own eyes. Thus visual patterns enhance mathematics' more cerebral tradition of variables and symbols . In this way, mathematics can appeal to more than one of students' multiple intelligences.

- To connect mathematics with real life, instructors increasingly stress **authentic problems** which, in contrast to the stylized word problems of traditional algebra, encourage exploration of realistic tasks for which mathematical relationships can provide significant insight. Of course real data will be messy data, obscuring the perfect patterns that arise from mathematical formulas. By exploring patterns and relationships throughout the K-12 curriculum, both in applied and theoretical contexts, students will see mathematics not just as an artificial game with symbols, but as a powerful tool for use with real problems.

> "Problems students are expected to solve should not only relate to the real world but also should be something people would realistically encounter on the job or in their role as citizens or parents."
> (Packer, 1997 p. 150.)

Patterns not only engage students in all areas of mathematics (arithmetic, algebra, geometry, probability, statistics, etc.), but also in all types of mathematical activity:

- *Recognize:* Discover mathematical opportunities in diverse contexts;
- *Visualize*: See patterns in data and in non-mathematical situations;
- *Verbalize:* Express in words the nature of patterns seen with the eye;
- *Symbolize:* Formalize in mathematical symbols the relationships found in patterns;
- *Analyze:* Relate one pattern to another, and predict new patterns.

Patterns help students experience not only the power of mathematics to solve problems but also the power of logic to validate mathematical statements. In a world darkened by a steady rain of conflicting claims, the possibility of clarity afforded by algebraic reasoning shines like a beacon of rationality. Patterns, relationships, and functions thus carry on the historic mission of algebra in the schools: to help students learn to think, and to provide them with the tools to solve important problems.

> "In mathematics, reasoning is the standard of truth." (MSEB, 1990, p. 47.)

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