Data, Shapes, Symbols: Achieving Balance in School Mathematics

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Mathematics is our "invisible culture" (Hammond 1978). Few people have any idea how much mathematics lies behind the artifacts and accoutrements of modern life. Nothing we use on a daily basis—houses, automobiles, bicycles, furniture, not to mention cell phones, computers, and Palm Pilots—would be possible without mathematics. Neither would our economy nor our democracy: national defense, Social Security, disaster relief, as well as political campaigns and voting, all depend on mathematical models and quantitative habits of mind.

Mathematics is certainly not invisible in education, however. Ten years of mathematics is required in every school and is part of every state graduation test. In the late 1980s, mathematics teachers led the national campaign for high, publicly visible standards in K-12 education. Nonetheless, mathematics is the subject that parents most often recall with anxiety and frustration from their own school experiences. Indeed, mathematics is the subject most often responsible for students' failure to attain their educational goals. Recently, mathematics curricula have become the subject of ferocious debates in school districts across the country.

My intention in writing this essay is to make visible to curious and uncommitted outsiders some of the forces that are currently shaping (and distorting) mathematics education. My focus is on the second half of the school curriculum, grades 6 to 12, where the major part of most students' mathematics education takes place. Although mathematics is an abstract science, mathematics education is very much a social endeavor. Improving mathematics education requires, among many other things, thorough understanding of the pressures that shape current educational practice. Thus I begin by unpacking some of the arguments and relevant literature on several issues—tracking, employment, technology, testing, algebra, data, and achievement—that are responsible for much of the discord in current public discussion about mathematics education.

Following discussion of these external forces, I examine the changing world of mathematics itself and its role in society. This leads to questions of context and setting, of purposes and goals, and quickly points in the direction of broader mathematical sciences such as statistics and numeracy. By blending the goals of mathematics, statistics, and numeracy, I suggest—in the final section of the essay—a structure for mathematics education in grades 6 to 12 that can help more students leave school equipped with the mathematical tools they will need for life and career.

External Forces

Beginning with *A Nation at Risk* (National Commission on Excellence in Education 1983) and continuing through *Before It's Too Late,* the report of the Glenn Commission (National Commission on Mathematics and Science Teaching for the 21st Century 2000), countless hand-wringing reports have documented deficiencies in mathematics education. Professional societies (American Mathe-

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matical Association of Two-Year Colleges 1995; National Council of Teachers of Mathematics 1989, 2000) have responded with reform-oriented recommendations while states (e.g., California, Virginia, Minnesota, Texas, and dozens of others) have created standards and frameworks suited to their local traditions. Analysis of these proposals, much of it critical, has come from a wide variety of sources (e.g., Cheney 1997; Kilpatrick 1997; Wu 1997; Raimi and Braden 1998; Gavosto, et al. 1999; Stotsky 2000). In some regions of the country, these debates have escalated into what the press calls "math wars" (Jackson 1997).

Nearly one-hundred years ago, Eliakim Hastings Moore, president of the young American Mathematical Society, argued that the momentum generated by a more practical education in school would better prepare students to proceed "rapidly and deeply" with theoretical studies in higher education (Moore 1903). In the century that followed, mathematics flowered in both its practical and theoretical aspects, but school mathematics bifurcated: one stream emphasized mental exercises with little obvious practical value; the other stream stressed manual skills with no theoretical value. Few schools ever seriously followed Moore's advice of using practical education as a stepping-stone to theoretical studies.

Now, following a century of steady growth based on rising demand and a relatively stable curricular foundation, a new president of the American Mathematical Society has warned his colleagues that the mathematical sciences are undergoing a "phase transition" from which some parts might emerge smaller and others dispersed (Bass 1997). The forces creating this transition are varied and powerful, rarely under much control from educators or academics. I have selected only a few to discuss here, but I believe these few will suffice to illustrate the nuances that too often are overlooked in simplistic analyses of editorials, op-ed columns, and school board debate. I begin with the contentious issue of tracking.

Tracking

Until quite recently, mathematics was never seen as a subject to be studied by all students. For most of our nation's history, and in most other nations, the majority of students completed their school study of mathematics with advanced arithmetic—prices, interest, percentages, areas, and other topics needed for simple commerce. Only students exhibiting special academic interest studied elementary algebra and high school geometry; even fewer students, those exhibiting particular mathematical talent, took advanced algebra and trigonometry. For many generations, the majority of students studied only commercial or vocational mathematics, which contained little if any of what we now think of as high school mathematics.

In recent decades, as higher education became both more important and more available, the percentage of students electing the academic track increased substantially. In the 1970s, only about 40 percent of U.S. students took two years of mathematics (algebra and geometry) in secondary school; 25 years later that percentage has nearly doubled. The percentage of high school students taking three years of mathematics has climbed similarly, from approximately 30 percent to nearly 60 percent (National Science Board 1996; Dossey and Usiskin 2000).

This shift in the presumption of mathematics as a subject for an academic elite to mathematics as a core subject for all students represents the most radical transformation in the philosophy of mathematics education in the last century. In 1800, Harvard University expected of entering students only what was then called "vulgar" arithmetic. One century later, Harvard expected a year of Euclid; two centuries later—in 2000—Harvard expects that most entering students have studied calculus. In no other subject has the expected level of accomplishment of college-bound students increased so substantially. These changes signal a profound shift in public expectations for the mathematical performance of high school graduates, a change that is sweeping the globe as nations race to keep up with rapidly advancing information technology. Secondary school mathematics is no longer a subject for the few, but for everyone.

In response to the increasing need for mathematical competence in both higher education and the high-performance workplace, the National Council of Teachers of Mathematics (NCTM) initiated the 1990s movement for national standards by recommending that *all* students learn a common core of high-quality mathematics including algebra, geometry, and data analysis (NCTM 1989). Dividing students into academic and nonacademic tracks, NCTM argued, no longer makes the sense it once did when the United States was primarily an agrarian and assembly line economy. In this old system—remnants of which have not yet entirely disappeared—college-bound students were introduced to algebra and geometry while those in vocational tracks were expected only to master arithmetic. Because algebra was not needed in yesterday's world of work, it was not taught to students in the lower tracks. This vocational tradition of low expectations (and low prestige) is precisely what NCTM intended to remedy with its call for a single core curriculum for all students.

Yet even as "mathematics for all" has become the mantra of reform, schools still operate, especially in mathematics, with separate tracks as the primary strategy for delivery of curriculum. They are reinforced in this habit by teachers who find it easier to teach students with similar mathematical backgrounds and by parents who worry not that *all* children learn but that their *own* children learn. Indeed, parents' anxiety about ensuring their own children's success has rapidly transformed an academic debate about tracking into one of the more contentious issues in education (e.g., Oakes 1985; Oakes 1990; Sheffield 1999). Thus, the most common critique of the NCTM standards is that by advocating the same mathematics for all they fail to provide mathematically talented students with the stimulation they need and deserve (Jackson 1997; Wu 1996).

As the world of work has become increasingly quantitative, even the historic reasons for tracking have come under scrutiny. From advanced manufacturing to precision agriculture, from medical imaging to supermarket management, competitive industries now depend not just on arithmetic and percentages but also on such tools as quantitative models, statistical quality control, and computer-controlled machines. Effective vocational programs must now set demanding mathematics standards that reflect the same kinds of higher-order thinking heretofore found only in the academic track (Steen and Forman 1995). Although details of content differ, expectations for rigorous logical thinking are very similar.

Indeed, mathematics in the workplace offers students opportunities to grapple with authentic, open-ended problems that involve messy numbers, intricate chains of reasoning, and lengthy multistep solutions—opportunities that rarely are found in traditional college-preparatory mathematics curricula. By deploying elementary mathematics in sophisticated settings, modern work-based tasks can give students not only motivation and context but also a concrete foundation from which they can later abstract and generalize.

Both traditional tracks—academic and vocational—have been pushed by their clienteles to increase significantly the level of mathematical performance expected of students. To be sure, not every school or program has responded equally to these heightened expectations. There are still large numbers of students who complete a vocational program (and sometimes an academic program) without really mastering any significant part of secondary school mathematics. But the direction of change is clear and the movement to eliminate dead-end courses is gaining momentum.

Employment

During the last decade of the twentieth century, just as the movement for academic standards began, business and industry launched a parallel effort to articulate entry-level skill standards for a broad range of industries (NSSB 1998) as well as to suggest better means of linking academic preparation with the needs of employers (Bailey 1997; Forman and Steen 1998).

Although preparing students for work has always been one purpose of education, teachers generally adopt broader goals and more specifically academic purposes. Mathematics educators are no exception. The canonical curriculum of school mathematics—arithmetic, algebra, geometry, trigonometry, calculus—is designed primarily to introduce students to the discipline of mathematics and only incidentally to provide tools useful for jobs and careers. Were schools to design mathematics programs expressly for work and careers, the selection of topics, the order in which they are taken up, and the kinds of examples employed would be substantially different.

The contrast between these two perspectives—mathematics in school versus mathematics at work—is especially striking (Forman and Steen 1999). Mathematics in the workplace makes sophisticated use of elementary mathematics rather than, as in the classroom, elementary use of sophisticated mathematics. Work-related mathematics is rich in data, interspersed with conjecture, dependent on technology, and tied to useful applications. Work contexts often require multistep solutions to open-ended problems, a high degree of accuracy, and proper regard for required tolerances. None of these features is found in typical classroom exercises.

Even core subjects within mathematics change when viewed from an employment perspective. Numbers in the workplace are embedded in context, used with appropriate units of measurement, and supported by computer graphics. They are used not just to represent quantities but also to calculate tolerances and limit errors. Algebra is used not so much to solve equations as to represent complex relationships in symbolic form. Geometry is used not so much to prove results as for modeling and measuring, especially in three dimensions.

It should come as no surprise, therefore, to discover that employers are distressed by the weak mathematical and quantitative skills of high school graduates. It is not uncommon for employers in high-performance industries such as Motorola, Siemens, and Michelin to find that only 1 in 20 job applicants has the skills necessary to join their training programs, and that only 1 in 50 can satisfactorily complete job training. (This employment situation is the industrial face of the immense remediation problem facing colleges and universities.)

It turns out that what current and prospective employees lack is not calculus or college algebra, but a plethora of more basic quantitative skills that could be taught in high school but are not (Murnane and Levy 1996; Packer 1997). Employees need statistics and three-dimensional geometry, systems thinking and estimation skills. Even more important, they need the disposition to think through problems that blend quantitative data with verbal, visual, and mechanical information; the capacity to interpret and present technical information; and the ability to deal with situations when something goes wrong (MSEB 1995). Although many jobs in the new economy require advanced training in mathematics, most do not. Nonetheless, all require a degree of numeracy unheard of a generation earlier as computers, data, and numbers intrude into the language of ordinary work.

This broader perspective of employers is well expressed in an influential government report entitled *What Work Requires of Schools* (Secretary's Commission on Achieving Necessary Skills 1991). Instead of calling for *subjects* such as mathematics, physics, and history, this so-called SCANS report asks for *competencies* built on a foundation of basic skills (reading, writing, listening, speaking, arithmetic), thinking skills (creative thinking, reasoning, problem solving, decision making, processing symbols, acquiring and applying new knowledge), and personal qualities (responsibility, self-esteem, sociability, self-management, integrity). These competencies, similar to what in other countries are sometimes called "key skills," are:

- *Resources:* Time, money, material, facilities, and human resources
- *Interpersonal:* Teamwork, teaching, service, leadership, negotiation, and diversity
- *Information:* Acquire, evaluate, organize, maintain, interpret, communicate, and transform
- *Systems:* Understand, monitor, and improve social, organizational, and technological systems
- *Technology:* Select, apply, and maintain technology

Mathematical thinking is embedded throughout these competencies, not just in the set of basic skills but as an essential component of virtually every competency. Reasoning, making decisions, solving problems, managing resources, interpreting information, understanding systems, applying technology — all these and more build on quantitative and mathematical acumen. But they do not necessarily require fluency in factoring polynomials, deriving trigonometric identities, or other arcana of school mathematics (Packer, see pp. 39–41).

Technology

The extraordinary ability of computers to generate and organize data has opened up an entire new world to mathematical analysis. Mathematics is the science of patterns (Steen 1988; Devlin 1994) and technology enables mathematicians (and students) to study patterns as they never could before. In so doing, technology offers mathematics what laboratories offer science: an endless source of evidence, ideas, and conjectures. Technology also offers both the arts and sciences a new entrée into the power of mathematics: fields as diverse as cinema, finance, and genetics now deploy computer-based mathematical tools to discover, create, and explore patterns.

Modern computers manipulate data in quantities that overwhelm traditional mathematical tools. Computer chips now finally have achieved sufficient speed and power to create visual displays that make sense to the human eye and mind. Already visualization has been used to create new mathematics (fractals), to develop new proofs (of minimal surfaces), to provide tools for new inferences (in statistics), and to improve instruction (geometer's sketchpad). Indeed, the computer-enhanced symbiosis of eye and image is fundamentally changing what it means to understand mathematics.

Computers also are changing profoundly how mathematics is practiced. The use of spreadsheets for storing, analyzing, and displaying data is ubiquitous in all trades and crafts. So too are computer tools of geometry that enable projection, rotation, inversions, and other fundamental operations to be carried out with a few keystrokes. Scientists and engineers report that, for students in these fields, facility with spreadsheets (as well as other mathematical software) is as important as conceptual understanding of mathematics and more valuable than fluency in manual computation (Barker 2000). With rare exceptions (primarily theoretical scientists and mathematicians) mathematics in practice means mathematics mediated by a computer.

As the forces unleashed by the revolution in technology change the character of mathematics, so they also impact mathematics education. It has been clear for many years that technology alters priorities for mathematics education (e.g., MSEB 1990). Much of traditional mathematics (from long division to integration by parts) was created not to enhance understanding but to provide a means of calculating results. This mathematics is now embedded in silicon, so training people to implement these methods with facility and accuracy is no longer as important as it once was. At the same time, technology has increased significantly the importance of other parts of mathematics (e.g., statistics, number theory, discrete mathematics) that are widely used in information-based industries.

Calculators and computers also have had enormous—and controversial—impact on mathematics pedagogy. Wisely used, they can help students explore patterns and learn mathematics by direct experience (Hembree and Dessart 1992; Askew and William 1995; Waits and Demana 2000), processes heretofore only possible through tedious and error-prone manual methods. Unwisely used, they become an impediment to students' mastery of basic skills or, even worse, a device that misleads students about the true nature of mathematics. Students who rely inappropriately on calculators often confuse approximations with exact answers, thereby depriving themselves of any possibility of recognizing or appreciating the unique certainty of mathematical deduction.

In the long run, technology's impact on mathematics education may be much broader than merely influencing changes in content or pedagogy. The rapid growth of a technology-driven economy that creates wealth as much from information and ideas as from labor and capital magnifies enormously the importance of intellectual skills such as mathematics. It also increases the social costs of differential accomplishment in school mathematics. Because of technology, it matters much more now than previously if a student leaves school with weak mathematical skills.

At the same time, computers and calculators are increasing dramatically the number of people who use mathematics, many of whom are not well educated in mathematics. Previously, only those who learned mathematics used it. Today many people use mathematical tools for routine work with spreadsheets, calculators, and financial systems, tools that are built on mathematics they have never studied and do not understand. This is a new experience in human history, with problematic consequences that we are only gradually discovering.

Finally, as the technology-driven uses of mathematics multiply, pressure will mount on schools to teach both information technology and more and different mathematics (ITEA 2000; NRC 1999). At the same time, and for the same reasons, increasing pressure will be applied on teachers and schools to ensure that no child is left behind. Alarms about the "digital divide" already have sounded and will continue to ring loudly in the body politic (Compaigne 2001; Norris 2001; Pearlman 2002). The pressure on mathematics to form a bipartisan alliance with technology in the school curriculum will be enormous. This easily could lead to a new type of tracking—one track offering the minimal skills needed to operate the new technology with little if any understanding, the other offering mathematical understanding as the surest route to control of technology. Evidence of the emergence of these two new cultures is not hard to find.

Testing

Largely because of its strong tradition of dispersed authority and local control, the United States has no system to ensure smooth articulation between high school and college mathematics programs. Instead, students encounter a chaotic mixture of traditional and standards-based high school curricula; Advanced Placement (AP) examinations in Calculus, Statistics, and Computer Science; very different SAT and ACT college entrance examinations; diverse university admissions policies; skills-based mathematics placement examinations; and widely diverse first-year curricula in college, including several levels of high school algebra (elementary, intermediate, and "college") and of calculus ("hard" (mainstream), "soft," and "reformed").

This cacophony of tests and courses is not only confusing and inefficient but also devastating for students who lack the support of experienced adult advocates. Following the rules and passing the tests does not necessarily prepare students either for employment or for continuing education. As a consequence, many new graduates find that they "can't get there from here." For some students, mathematics education turns out to be a "hoax" (Education Trust 1999).

The negative consequences of this incoherent transition have been magnified greatly in recent years as states began, for the first time, to institute meaningful (high-stakes) exit examinations that students must pass to receive a high school diploma (Gardner 1999; Sacks 2000; Shrag 2000). Many states have been shocked by the low passing rates on such examinations, and have had to retrench on their graduation requirements (Groves 2000). At the same time, parents and politicians have increased their emphasis on tests such as the SAT, ACT, and AP that have significant influence in college admissions even as university officials (led by University of California President Richard Atkinson) have called into question the appropriateness of these tests as a gateway to college (Atkinson 2001).

Despite all this testing, once students arrive in college, hundreds of thousands find themselves placed in remedial courses such as intermediate algebra in which they are required to master arcane skills that rarely are encountered in adult life. As more students pursue postsecondary study—both before and while working—and as these students bring to their studies increasingly diverse backgrounds and career intentions, incoherent and arbitrary testing in the transition from school to college becomes increasingly untenable.

A rational system of mathematics education should provide clear and consistent messages about what knowledge and skills are expected at each educational level. Ideally, graduation examinations from secondary school also would certify, based on different scores, admission to college without remediation. Such a system would require that everyone involved in the transition from high school to college concur on the expected outcomes of high school mathematics and that these goals be reflected in the tests. To be politically acceptable, transition tests must be within reach of most students graduating from today's high schools, yet to be educationally useful, they must ensure levels of performance appropriate to life, work, and study after high school. No state has yet figured out how to meet both these objectives.

Algebra

In the Middle Ages, algebra meant calculating by rules (algorithms). During the Renaissance, it came to mean calculation with signs and symbols—using *x*'s and *y*'s instead of numbers. (Even today, laypersons tend to judge algebra books by the symbols they contain: they believe that more symbols mean more algebra, more words, less.) In subsequent centuries, algebra came to be primarily about solving equations and determining unknowns. School algebra still focuses on these three aspects: following procedures, employing letters, and solving equations.

In the twentieth century, algebra moved rapidly and powerfully beyond its historical roots. First it became what we might call the science of arithmetic—the abstract study of the operations of arithmetic. As the power of this "abstract algebra" became evident in such diverse fields as economics and quantum mechanics, algebra evolved into the study of *all* operations, not just the four found in arithmetic. Thus did it become truly the language of mathematics and, for that reason, the key to access in our technological society (Usiskin 1995).

Indeed, algebra is now, in Robert Moses' apt phrase, "the new civil right" (Moses 1995). In today's society, algebra means access. It unlocks doors to productive careers and democratizes access to big ideas. As an alternative to dead-end courses in general and commercial mathematics, algebra serves as an invaluable engine of equity. The notion that by identifying relationships we can discover things that are unknown—"that we can find out what *we* want to know"—is a very powerful and liberating idea (Malcolm 1997).

Not so long ago, high school algebra served as the primary filter to separate college-bound students from their work-bound classmates. Advocates for educational standards then began demanding "algebra for all," a significant challenge for a nation accustomed to the notion that only some could learn algebra (Steen 1992; Chambers 1994; Lacampagne et al. 1995; Silver 1997; NCTM and MSEB 1998). More recently, this clamor has escalated to a demand that every student complete algebra by the end of eighth grade (Steen 1999; Achieve 2001).

The recent emphasis on eighth-grade algebra for all has had the unfortunate side effect of intensifying distortions that algebra already imposes on school mathematics. One key distortion is an overemphasis on algebraic formulas and manipulations. Students quickly get the impression from algebra class that mathematics *is* manipulating formulas. Few students make much progress toward the broad goals of mathematics in the face of a curriculum dominated by the need to become fluent in algebraic manipulation. Indeed, overemphasis on algebra drives many students away from mathematics: most students who leave mathematics do so because they cannot see any value in manipulating strings of meaningless symbols.

What is worse, the focus on formulas as the preferred methodology of school mathematics distorts the treatment of other important parts of mathematics. For example, despite the complexity of its algebraic formula, the bell-shaped normal distribution is as ubiquitous in daily life as are linear and exponential functions and far more common than quadratic equations. As citizens, it is very helpful to understand that repeated measurements of the same thing (length of a table) as well as multiple measurements of different although similar things (heights of students) tend to follow the normal distribution. Knowing why some distributions (e.g., salaries, sizes of cities) do not follow this pattern is equally important, as is understanding something about the tails of the normal distribution—which can be very helpful in thinking about risks (or SAT scores).

Yet despite its obvious value to society, the normal distribution is all but ignored in high school mathematics, whereas quadratic and periodic functions are studied extensively. Many reasons can be advanced to explain this imbalance, e.g., that mathematicians favor models of the physical over the behavioral sciences. But surely one of the most important is that the algebraic formula for the normal distribution is quite complex and cannot be fully understood without techniques of calculus. The bias in favor of algebraic formulas as the preferred style of understanding mathematics—instead of graphs, tables, computers, or verbal descriptions—causes mathematics teachers to omit from the high school curriculum what is surely one of the most important and most widely used tools of modern mathematics.

That a subject that for many amounts to little more than rote fluency in manipulating meaningless symbols came to occupy such a privileged place in the school curriculum is something of a mystery, especially since so many parents, when they were students, found it unbearable. Perhaps more surprising is algebra's strong support among those many successful professionals who, having mastered algebra in school, found no use for it in their adult lives. Why is it that we insist on visiting on eighth graders a subject that, more than any other, has created generations of math-anxious and math-avoiding adults?

Many argue on the simple, pragmatic "civil right" ground that algebra is, wisely or unwisely, of central importance to the current system of tests that govern the school-to-college transition (not to mention providing essential preparation for calculus, which itself has taken on exaggerated significance in this same transition). But this is just a circular argument. We need to study algebra to pass tests that focus on algebra. And why do the tests focus on algebra? Because it is the part of mathematics that virtually all students study.

Others may cite, as grounds for emphasizing algebra, the widespread use of formulas in many different fields of work; however, this use is only a tiny part of what makes up the school subject of algebra. Moreover, most business people give much higher priority to statistics than to algebra. Some mathematicians and scientists assert that algebra is *the* gateway to higher mathematics, but this is so only because our curriculum makes it so. Much of mathematics can be learned and understood via geometry, or data, or spreadsheets, or software packages. Which subjects we emphasize early and which later is a choice, not an inevitability.

Lurking behind the resurgent emphasis on algebra is a two-edged argument concerning students who are most likely to be poorly educated in mathematics—poor, urban, first generation, and minority. Many believe that such students, whose only route to upward mobility is through school, are disproportionately disadvantaged if they are denied the benefits that in our current system only early mastery of algebra can confer. Others worry that emphasis on mastering a subject that is difficult to learn and not well taught in many schools will only exacerbate existing class differences by establishing algebra as a filter that will block anyone who does not have access to a very strong educational environment. Paradoxically, and unfortunately, both sides in this argument appear to be correct.

Data

Although algebra and calculus may be the dominant goals of school mathematics, in the real world mathematical activity usually begins not with formulas but with data. Measurements taken at regular intervals—be they monthly sales records, hourly atmospheric pressure readings, or millisecond samples of musical tones—form the source data for mathematical practice. Rarely if ever does nature present us with an algebraic formula to be factored or differentiated. Although the continuous model of reality encapsulated by algebra and calculus is a powerful tool for developing theoretical models, real work yielding real results must begin and end in real data.

In past eras, mathematics relied on continuous models because working with real data was too cumbersome. An algebraic or differential equation with three or four parameters could describe reasonably well the behavior of phenomena with millions of potential data points, but now computers have brought digital data into the heart of mathematics. They enable practitioners of mathematics to work directly with data rather than with the simplified continuous approximations that functions provide. Moreover, they have stimulated whole new fields of mathematics going under names such as combinatorics, discrete mathematics, and exploratory data analysis.

Thus as school mathematics has become increasingly preoccupied with the role of algebra, many users of mathematics have discovered that combinatorial and computer methods are of far greater utility. Whereas school algebra deals primarily with models and continuous functions, combinatorics and data analysis deal with measurements and discrete data. The one reflects a Platonic world of ideal objects, the other the realism of measured quantities. In the Platonic world, theorems are eternal; in the real world, computations are contingent. This contrast between the ideal and the utilitarian can be seen from many different perspectives ranging from philosophical to pedagogical.

One such domain is education. The competition for curricular time between functions and data reflects fundamental disagreements about the nature of mathematics as a discipline and as a school subject. Traditionally, and philosophically, mathematics has been thought of as a science of ideal objects—numbers, quantities, and shapes that are precisely defined and thus amenable to logically precise relations known as theorems. In practice, mathematics presents a more rough-and-ready image: it is about solving problems in the real world that involve measured quantities that are never perfectly precise. Tension between these two views of mathematics has a long history. But now, with the advent of computers, this tension has resurfaced with even greater force and significance. At its core, the debate is about the definition of mathematics as a discipline.

Achievement

Strained by a growing number of forces and pressures (only some of which are discussed here), U.S. mathematics educators have found it very difficult to improve student achievement—education's bottom line. For at least the last half-century, graduates of U.S. secondary schools have lagged behind their peers in other nations, especially those of the industrial world and the former Communist bloc. Documentation of this deficiency has been most consistent in mathematics and science, subjects that are relatively common in the curricula of other nations and that are examined internationally at regular intervals. Some U.S. analysts seek to explain (or excuse) poor U.S. performance by hypothesizing a negative impact of our relatively heterogeneous population, or conjecturing that a larger percentage of U.S. students complete secondary school, or arguing that other nations (or the United States) did not test a truly random sample. But despite these exculpatory claims, a central stubborn fact remains: on international tests administered over several decades to similarly educated students, the mathematics performance of U.S. eighth- and twelfth-grade students has always been well below international norms.

The most recent headlines came from TIMSS, the Third International Mathematics and Science Study, and its repeat, TIMSS-R.

The TIMSS results, confirmed by TIMSS-R, document a decline in the performance of U.S. students, as compared with their peers in other nations, as they progress through school (IEA 2000). Fourth graders in the United States have command of basic arithmetic on a par with students in most other nations, but the longer U.S. students study mathematics, the worse they become at it, comparatively speaking (Beaton et al. 1996; Schmidt et al. 1996). Middle school mathematics, especially, exhibits "a pervasive and intolerable mediocrity" (Silver 1998) that sends students into a downward glide that leaves them in twelfth grade with a mathematical performance that is virtually at the bottom of all industrialized nations. Even the best U.S. twelfth-grade students who are enrolled in advanced mathematics courses perform substantially below the average twelfth-grade students in most other nations (National Center for Education Statistics 1998).

The TIMSS findings are consistent with other analyses of U.S. student achievement from an international perspective (McKnight et al. 1987; Lapointe et al. 1989). They document the consequences of a leisurely curriculum in the last half of elementary school when textbooks fail to introduce much that has not already been covered (Flanders 1987). What makes matters even worse is the long-standing performance differences among white, black, and Hispanic students at all grade levels (Campbell et al. 1996; National Center for Education Statistics 2000). Although this performance gap has been narrowing on tasks that assess procedural knowledge and skills, substantial differences remain on tasks that assess conceptual understanding, mathematical reasoning, and problem solving (Secada 1992; Kenney and Silver 1996). Thus at a time of increasing integration of a global economy, large numbers of U.S. students, disproportionately minority, leave school significantly behind world norms in the language of the information age—mathematics.

Not surprisingly, more detailed examination of the TIMSS results reveals that U.S. students perform relatively better on some mathematical topics and worse on others. For example, relative to their international peers, our eighth-grade students are especially weak in geometry, measurement, and proportional reasoning, although closer to average in arithmetic and algebra. A similar profile emerged from the Second International Mathematics Study (SIMS) conducted in 1981–82 (Crosswhite et al. 1986; McKnight et al. 1987). Interestingly, the topics on which our students lag behind international norms (for example, measurement, geometry, and proportional thinking) are precisely the areas cited by noneducators as most important for adult life.

Quantitative Practices

The forces created by differential tracking, needs of employment, impacts of technology, misaligned testing, overemphasis on algebra, underemphasis on data, and student underachievement exert profound influence on schools, teachers, and students. These forces shape and often distort the educational process, constraining teachers and enticing students in directions that are rarely well aligned with sound educational goals. To have a significant and lasting effect, changes proposed for school mathematics must take these external forces into account, seeking wherever possible to use them for advantageous leverage.

School mathematics also needs to be responsive to changes in mathematics itself—its scope, practice, methods, and roles in society. Most people think of mathematics as unchanging, as a collection of formulas and facts passed down like ancient texts from earlier generations. Nothing could be further from the truth. Mathematical discovery has grown at an amazing rate throughout the past century, accelerating in recent decades as computers provide both new problems to solve and new tools with which to solve old problems (Odom 1998). At the same time, and for much the same reason, the roles played by mathematics in society have expanded at a phenomenal rate. No longer confined to specialized fields such as engineering or accounting, mathematical methods permeate work and life in the information age.

As mathematics has expanded rapidly to provide models for computer-based applications, so too has statistics, the science of data. Statistics is a hybrid discipline with some roots in mathematics but even more in social science, agriculture, government records, economic policy, and medical research. Especially since 1989, when NCTM called for greater emphasis on statistics and data analysis in the school curriculum, statistics has become part of the agenda for school mathematics. Arguably, for most students it may be the most important part.

As mathematical ideas increasingly permeate public policy, those concerned with citizenship and democracy have begun to see a real need for quantitative practices not readily subsumed by either mathematics or statistics. These practices, called "numeracy" elsewhere, are relatively new in the U.S. educational context. Indeed, the explicit parallels between numeracy and literacy as marks of an educated person are really no more than 10 or 15 years old (Paulos 1988; Steen 1990). As the disciplines of mathematics and statistics have expanded in scope and influence, their impact on public life has created a rising demand for the interdisciplinary (or cross-cutting) capacity we call numeracy. Mathematics and numeracy are two sides of the same coin—the one Platonic, the other pragmatic, the one abstract, the other contextual.

As all three forms of quantitative practice—mathematics, statistics, and numeracy—evolve under the selective pressures of information technology and a global economy, schools must find ways to teach all three. Before exploring how this might be done, we first elaborate on the nature of these three domains of quantitative practice.

Mathematics

During the last half-century, as mathematics in school grew from an elite to a mass subject, mathematics expanded into a portfolio of mathematical sciences that now includes, in addition to traditional pure and applied mathematics, subjects such as statistics, financial mathematics, theoretical computer science, operations research (the science of optimization) and, more recently, financial mathematics and bioinformatics. (It is a little-appreciated fact that most of the advances—and fortunes—being made in investments, genetics, and technology all derive from clever applications of sophisticated mathematics.) Although each of these specialties has its own distinctive character, methodologies, standards, and accomplishments, they all build on the same foundation of school and college mathematics.

Mathematics is far more than just a tool for research. In fact, its most common uses—and the reason for its prominent place in school curricula—are routine applications that are now part of all kinds of jobs. Examples include:

- Testing products without destroying them
- Managing investments to minimize risks while maximizing returns
- Creating terrain maps for farmers that reflect soil chemistry and moisture levels
- Processing photographic images to transform, clarify, and combine
- Detecting disease by monitoring changes in medical images and data
- Creating special cinematic effects such as moving clouds and rushing water
- Anticipating changes in production processes
- Controlling risks by managing distribution of hazardous materials
- Designing products to minimize costs of construction, maintenance, and operation
- Interpreting vital signs displayed as dynamic graphs of biological data
- Minimizing total costs of materials, inventory, time, shipments, and waste

If we look at these common uses of mathematics from the perspective of the school curriculum, we see that mathematics at work is very different from mathematics in school:

- Arithmetic is not just about adding, subtracting, multiplying, and dividing but about units and conversions, measurements and tolerances, spreadsheets and calculators, and estimates and accuracy.
- Numbers are not just about place value and digits but about notation and coding, index numbers and stock market averages, and employment indexes and SAT scores.
- Geometry is not just about the properties of circles, triangles, areas, and volumes but about shapes and measurements in three dimensions, reading maps and calculating latitude and longitude, using dimensions to organize data, and modern tools such as global positioning systems (GPS) and geographic information systems (GIS).
- Statistics is not just about means, medians, and standard deviations but about visual displays of quantitative ideas (for example, scatter plots and quality control charts) as well as random trials and confidence intervals.
- Logic is not just about mathematical rigor and deductive proof but about hypotheses and conjectures, causality and correlation, and random trials and inference in the face of incomplete information.
- Probability is not just about calculating combinations but about estimating and comparing risks (for example, of accidents, diseases, or lotteries) as well as about chance and randomness (in coincidences or analyses of bias claims).
- Applications are not just about solving word problems but about collecting, organizing, and interpreting data; allocating resources and negotiating differences; and understanding annuities and balancing investments.
- Proof is not just about logical deduction but about conjectures and counterexamples, scientific reasoning and statistical inference, and legal standards such as preponderance of evidence or beyond reasonable doubt.
- Technology is not just about doing arithmetic, performing algebra, or creating graphs but about facility with spreadsheets, statistical packages, presentation software, and Internet resources.

Mathematics in practice is far subtler than mathematics in school. Elementary mathematical ideas applied in sophisticated settings are amazingly powerful but rarely appreciated. An important conclusion from this examination of mathematics in practice is that topics common to school mathematics have surprising depth and power in their own right, quite apart from their role in providing prerequisites for college mathematics. Indeed, one could productively pursue applications of school topics for several years without ever taking up the more abstract concepts of calculus (or even so-called "college" algebra). But no one does this, preferring instead to rush students as quickly as possible to the abstractions of calculus.

Statistics

The age of information is an age of numbers. We are surrounded by data that both enrich and confuse our lives. Numbers provide descriptions of daily events, from medical reports to political trends and social policy. News reports are filled with charts and graphs, while politicians debate quantitatively based proposals that shape public policy in education, health, and government.

The study of numbers is usually associated with statistics. In schools, the term "quantitative literacy" is often employed as an informal synonym for "elementary statistics." Although statistics is today a science of numbers and data, historically (and etymologically) it is the science of the state that developed in the Napoleonic era when central governments used data about population, trade, and taxes to assert control over distant territory. The value of systematic interpretation of data quickly spread to agriculture, medicine, economics, and politics. Statistics now underlies not only every economic report and census but also every clinical trial and opinion survey in modern society.

However valuable statistics may be, it seems never to have shared the curricular privilege accorded to mathematics. Indeed, high school mathematics devotes relatively little emphasis to topics designed to build a numbers-based bridge from the arithmetic of the elementary grades to the subtle and fascinating world of data and statistics. Computers have significantly transformed the potential, power, and pedagogy of statistics (Hoaglin and Moore 1992) and this evolution has profoundly changed the relation between mathematics, statistics, and their many client disciplines (Moore and Cobb 2000). It is past time for statistics to claim its proper place in the school mathematics curriculum.

One impediment statistics faces is a public perception that it is not as rigorous as calculus. This perception is no doubt due to its association with the "soft" sciences of psychology and economics, in contrast with the "hard" calculus-based disciplines of physics and engineering. Evidence from the new and rapidly growing AP Statistics course, however, confirms what many teachers have long known—that the subtle reasoning involved in data-based statistical inference is harder for students to grasp and explain than the comparable symbol-based problems and proofs in a typical calculus course. Properly taught, statistics is probably a better vehicle than algebra and calculus for developing students' capacity to reason logically and express complex arguments clearly.

Statistics is also very practical; far more so than any part of the algebra-trigonometry-calculus sequence that dominates school mathematics. Every issue in the daily newspaper, every debate that citizens encounter in their local communities, every exhortation from advertisers invites analysis from a statistical perspective. Statistical reasoning is subtle and strewn with counterintuitive paradoxes. It takes a lot of experience to make statistical reasoning a natural habit of mind (Nisbett et al. 1987; Hoffrage et al. 2000). That is why it is important to start early and to reinforce at every opportunity.

Numeracy

The special skills required to interpret numbers—what we call numeracy or quantitative literacy—are rarely mentioned in national education standards or state frameworks. Nonetheless, these skills nourish the entire school curriculum, including not only the natural, social, and applied sciences but also language, history, and fine arts (Steen 1990). They parallel and enhance the skills of literacy—of reading and writing—by adding to words the power of numbers.

Numeracy lies at the intersection of statistics, mathematics, and democracy. Like statistics, numeracy is centered on interpretation of data; like mathematics, numeracy builds on arithmetic and logic. But the unique niche filled by numeracy is to support citizens in making decisions informed by evidence. Virtually every major public issue—from health care to Social Security, from international economics to welfare reform—depends on data, projections, inferences, and the kind of systematic thinking that is at the heart of quantitative literacy. So too do many aspects of daily life, from selecting telephone services to buying a car, from managing household expenses to planning for retirement. For centuries, verbal literacy has been recognized as a free citizen's best insurance against ignorance and society's best bulwark against demagoguery. Today, in the age of data, numeracy joins literacy as the guarantor of liberty, both individual and societal (Steen 1998, 2000).

Numeracy is largely an approach to thinking about issues that employs and enhances both statistics (the science of data) and mathematics (the science of patterns). Yet unlike statistics, which is primarily about uncertainty, numeracy is often about the logic of certainty. And unlike mathematics, which is primarily about a Platonic realm of abstract structures, numeracy often is anchored in data derived from and attached to the empirical world. Surprisingly to some, this inextricable link to reality makes quantitative reasoning every bit as challenging and rigorous as mathematical reasoning.

Mathematics teachers often resist emphasizing data because the subject they are trying to teach is about Platonic ideals—numbers and functions, circles and triangles, sets and relationships. Employers and parents, however, often are frustrated by this stance because school graduates so frequently seem inexperienced in dealing with data, and the real world presents itself more often in terms of data than in the Platonic idealizations of mathematics.

Although numeracy depends on familiar mathematical topics from arithmetic, algebra, and geometry, its natural framework is commonly described in broader terms (Steen 2001). Some are foundational, focused on learned skills and procedures:

- *Practical Skills:* Using elementary mathematics in a wide variety of common situations
- *Confidence with Mathematics:* Being comfortable with numbers and at ease in applying quantitative methods
- *Number Sense:* Estimating with confidence; employing common sense about numbers; exhibiting accurate intuition about measurements
- *Mathematics in Context:* Using mathematical tools in settings in which the context provides both meaning and performance expectations
- *Prerequisite Knowledge:* Using a wide range of algebraic, geometric, and statistical tools that are required for many fields of postsecondary education

Other elements of numeracy live on a higher cognitive plateau and represent capacities as useful and ingrained as reading and speaking:

- *Interpreting Data:* Reasoning with data, reading graphs, drawing inferences, and recognizing sources of error
- *Making Decisions:* Using logical and quantitative methods to solve problems and make decisions in everyday life
- *Symbol Sense:* Employing, reading, and interpreting mathematical symbols with ease; exhibiting good sense about their syntax and grammar
- *Thinking Logically:* Analyzing evidence, reasoning carefully, understanding arguments, questioning assumptions, detecting fallacies, and evaluating risks
- *Cultural Appreciation:* Understanding the nature and history of mathematics, its role in scientific inquiry and technological progress, and its importance for comprehending issues in the public realm

Whereas the mathematics curriculum historically has focused on school-based knowledge, numeracy involves mathematics acting in the world. Typical numeracy challenges involve real data and uncertain procedures but require primarily elementary mathematics. In contrast, typical school mathematics problems involve simplified numbers and straightforward procedures but require sophisticated abstract concepts. The test of numeracy, as of any literacy, is whether a person naturally uses appropriate skills in many different contexts.

School Mathematics

For various reasons having to do with a mixture of classical tradition and colonial influence, the school curriculum in mathematics is virtually the same all over the world. Fifteen years ago, the secretary of the International Commission on Mathematics Instruction reported that apart from local examples, there were few significant differences to be found in the mathematics textbooks used by different nations around the world (Howson and Wilson 1986). Even a country as culturally separate as Japan follows a canonical "western" curriculum with only minor variations (Nohda et al. 2000). Detailed review of U.S. practice in the mid-1980s showed little significant change from the practice of previous decades (Hirsch and Zweng 1985). At the end of the twentieth century, therefore, a bird's-eye view of school mathematics reveals little substantive variation in either time or space.

Not surprisingly, however, a more refined analysis prepared in advance of the TIMSS study reveals subtle differences in scope, sequence, and depth (Howson 1991). The TIMSS study itself included an extensive analysis of curricula (and of teaching practices) in participating nations. This analysis showed significant variation in the number of topics covered at different grade levels, a variation that appears to be inversely correlated with student performance (Schmidt et al. 1997). In the case of mathematics education, it seems, more really is less: too many topics covered superficially lead to less student learning. The consensus of experts who have studied both domestic and international assessments is that neither the mathematics curriculum nor the classroom instruction is as challenging in the United States as it is in many other countries (e.g., Stevenson 1998).

This tradition of mathematics programs that are a “mile wide and an inch deep” is not easy to change. In contrast with most nations whose central ministries of education prescribe the goals and curriculum of school mathematics, the United States has no legally binding national standards. That is not to say, however, that we do not have a national curriculum. Textbooks, traditions, and standardized tests do as much to constrain mathematics teaching in the United States as national curricula do in other nations. All too often, these constraints produce what analysts of SIMS called an “underachieving” curriculum (McKnight et al. 1987).

In response to SIMS, NCTM prepared an innovative set of standards for school mathematics—a “banner” for teachers to rally behind in a national crusade to raise classroom expectations and student performance (NCTM 1989). Ten years later, NCTM revised these standards, producing a more tightly focused set of goals to guide states and districts as they developed their own frameworks and curriculum guides (NCTM 2000). This revised document, entitled *Principles and Standards for School Mathematics* (PSSM), is organized around five so-called “content” standards (number and operations, algebra, geometry, measurement, and data analysis and probability) and five “process” standards (problem solving, reasoning and proof, communication, connections, and representation).

The first five PSSM standards (see Appendix I) correspond to topics and chapter titles found in most mathematics textbooks. They represent the traditional content of mathematics: numbers, symbols, functions, shapes, measurements, probability, and the like. The second five, interestingly, fit better with the skills employers seek or the numeracy that citizenship requires—e.g., evaluating arguments, communicating quantitative ideas, interpreting real-world phenomena in mathematical terms. This distinction resonates with what we often hear from users of mathematics: it is not so much the specific content of mathematics that is valuable as the process of thinking that this content represents. Only mathematicians and mathematics teachers really worry much about the specifics of content.

Were all ten NCTM standards stressed equally in each grade from 6 to 12, and enriched with significant real-world examples, many more students would emerge from high school well prepared in mathematics, statistics, and numeracy. But this is far from true of today’s curricula. As assessment data show, the content goals of arithmetic and algebra are stressed at the expense of geometry, measurement, data analysis, and probability. In terms of the process goals, only problem solving is consistently stressed; the others—reasoning and proof, communication, connections, and representation—are barely visible in the curriculum and totally absent from common standardized tests. Some of this imbalance reflects differences in the cost of assessment: we test not what is most important but what is easiest and cheapest to test. The preoccupation with algebraic symbol manipulation is one result of this approach, because scoring mindless exercises is so much cheaper than judging thoughtful and unpredictable responses.

Challenges

Fixing school mathematics requires attention to many significant (and overwhelming) issues such as teacher competence, recruitment, salaries, and performance; class size and classroom conditions; alignment of standards with textbooks and tests; and consistent support by parents, professionals, and politicians. Here I merely acknowledge these issues but do not deal with any of them.

Instead, my primary purpose in this essay is to think through the goals of mathematics in grades 6 to 12 in light of the significant forces that are shaping the environment of school mathematics. These include:

- Underperformance of U.S. students, especially in areas of mathematics that are seriously neglected in school instruction;
- Continued support for tracking in an environment in which all students need high-quality mathematical experiences;
- Employers’ demand for performance competencies that cut across academic areas;
- Changes in curricular priorities, pedagogical strategies, and career options due to the increasing mathematical power of technology;
- Inconsistent expectations and misaligned tests that confront students as they finish high school and move on to postsecondary education;
- Unprecedented increases in routine uses of mathematics and in the types of mathematics being used;
- Extraordinary expansion in sophisticated applications of elementary mathematics;
- Increasing reliance on inferences from numerical evidence in business decisions, analyses, and political debates;
- Rapid growth in the use of computer-generated data, graphs, charts, and tables to present information; and
- Confusion about the relative importance of algebra as one among many mathematical subjects that students must learn.

These environmental forces are not hidden. Everyone who is concerned about the quality of mathematics education is aware of them. Nonetheless, school mathematics continues to serve primarily as a conveyor belt to calculus that educates well only a minority of students. Many individuals and organizations have developed proposals for change (e.g., California Academic Standards Commission 1997; MSEB 1998; NCTM 2000; Achieve 2001; Steen 2001), but these proposals represent contrasting rather than consensus visions of school mathematics.

The traditional curriculum in grades 6 to 12 is organized like a nine-layer cake: advanced arithmetic, percentages and ratios, elementary algebra, geometry, intermediate algebra, trigonometry, advanced algebra, pre-calculus, and (finally) calculus. Each subject builds on topics that precede it, and each topic serves as a foundation for something that follows. Although this sequence has the benefit of ensuring (at least on paper) that students are prepared for each topic by virtue of what has come before, the sequence does this at the expense of conveying a biased view of mathematics (because topics are stressed or ignored primarily on the basis of their utility as a tool in calculus) and creating a fragile educational environment (because each topic depends on mastery of most preceding material). The inevitable result can be seen all around us: most students drop out of mathematics after they encounter a first or second roadblock, while many of those who survive emerge with a distorted (and often negative) view of the subject.

The intense verticality of the current mathematics curriculum not only encourages marginal students to drop out but also creates significant dissonance as states begin to introduce high-stakes graduation tests. Inevitably, student performance spreads out as students move through a vertical curriculum because any weakness generates a cascading series of problems in subsequent courses. The result is an enormous gap between curricular goals and a politically acceptable minimum requirement for high school graduation. Consequently, in most states, the only enforced mathematics performance level for high school graduation is an eighth- or ninth-grade standard. This large discrepancy between goals and achievement discredits mathematics education in the eyes of both parents and students.

Breadth and Connectedness

I suggest that the way to resolve these conflicts—and to address many of the environmental factors mentioned above—is to structure mathematics in grades 6 to 12 to stress breadth and connectedness rather than depth and dependency. Instead of selecting topics for their future utility, as prerequisites for something to follow that most students will never see, select topics for their current value in building linkages both within mathematics and between mathematics and the outside world. Instead of selecting topics for their contribution to the foundation of calculus, only one among many important parts of advanced mathematics, select topics for their contribution to a balanced repertoire of all the mathematical sciences. And in each grade, but especially in middle school, stress topics that contribute simultaneously to mathematics, statistics, and numeracy.

A good place to start is with the revised NCTM standards (Appendix I). These ten standards, if treated with equal seriousness and supplemented with significant connections to the real world, would provide a very strong framework for mathematics in grades 6 to 12. Unfortunately, "equal seriousness" is rare. Geometry, measurement, data analysis, and probability need as strong a presence in the curriculum as algebra and number. Similarly, reasoning, communication, and connections need as much emphasis as problem solving and representation.

There is an ever-present danger that these NCTM goals will be viewed as mere rhetoric and that not much will change in the actual priorities of teachers or in the tests that districts and states use to monitor student performance. Taking all the standards seriously means that students need to work with data as much as with equations, with measurements and units as much as with abstract numbers. To learn to communicate quantitatively, students need as much experience reading texts that use quantitative or logical arguments as they have with literary or historical texts. And they need experience not only with the self-contained exercises in mathematics textbooks but also with realistic problems that require a combination of estimation, assumption, and analysis.

To some, these broad goals may seem to move well beyond the security zone of objective, Platonic mathematics in which proof and precision matter most and transformation of symbols replaces narrative explanations as a means of expressing thought. They do indeed move well beyond this protected arena, into the pragmatic world of mathematical practice broadly conceived. Yet it is only in this broad domain, not in the more restricted sphere of symbolic thinking, that mathematics can assert its warrant to special status in the school curriculum. (If it makes purists feel better, perhaps this curriculum should be identified, as the profession is, by the term "mathematical sciences.")

To accomplish such a transformation, mathematics teachers must become diplomats, recruiting allies from teachers in other fields who will stress the role of mathematics in the subjects they teach. Mathematics can be seen both as a service subject (Howson et al. 1988) and as a subject served. Art abounds with geometry; history with data and probability; music with ratios and series; science with measurement and algebra; economics with data and graphs. Every subject relies on, and teaches, the NCTM process standards such as reasoning, communication, and problem solving. To build breadth and secure connections, the mathematical sciences must be taught both in the mathematics classroom and in classrooms across the entire curriculum (Steen 1997; Wallace 1999).

Middle Grades

For several reasons, it is helpful to think of the seven years of grades 6 to 12 in three parts: the middle grades 6 to 8; the core high school grades 9 to 11; and the transition grade 12. To oversimplify (but not by much), the goal for grades 6 to 8 would be numeracy, for grades 9 to 11, mathematical sciences, and for grade 12, options. Data analysis, geometry, and algebra would constitute three equal content components in grades 6 to 8 and in grades 9 to 11. (In this simplified synopsis, measurement and probability can be viewed as part of data analysis, while number and operations can be viewed as part of algebra; discrete mathematics and combinatorics are embedded in every topic.) The five NCTM process standards cut across all topics and grade levels, but rather than being left to chance, they do need to be covered intentionally and systematically.

Careful planning can ensure that the foundational parts of school mathematics are covered in grades 6 to 8, without tracking but with multiple points of entry and many opportunities for mutual reinforcement. There are many different ways to do this, one of which is being developed by a dozen or so states belonging to the Mathematics Achievement Partnership (Achieve 2001). In a curriculum designed for breadth and connections, anything not learned the first time will appear again in a different context in which it may be easier to learn. For example, graphing data gathered through measurement activities provides review of, or introduction to, algebra and geometry; finding lengths and angles via indirect measurements involves solving equations; and virtually every task in data analysis as well as many in algebra and geometry reinforces and extends skills involving number and calculation.

Used this way, with intention and planning, linkages among parts of mathematics can be reinforcing rather than life-threatening. Instead of leading to frustration and withdrawal, a missing link can lead to exploration of alternative routes through different parts of mathematics. If middle school teachers give priority to topics and applications that form the core of quantitative literacy, students will encounter early in their school careers those parts of mathematics that are most widely used, most important for most people, and most likely to be of interest. More specialized topics can and should be postponed to grades 9 to 11.

Secondary School

In high school, all students should take three additional years of mathematics in grades 9 to 11, equally divided between data analysis, geometry, and algebra but not sequentially organized. Parallel development is essential to build interconnections both within the mathematical sciences and with the many other subjects that students are studying at the same time. Parallel does not necessarily mean integrated, although there certainly could be integration in particular curricula. It does mean that in each grade, students advance significantly in their understanding of each component of the triad of data analysis, geometry, and algebra. Parallel development reduces the many disadvantages of the intense and unnecessary verticality.

The content of this curriculum would not differ very much from the recommendations in the NCTM standards. The core of mathematics—data analysis, geometry, and algebra—is what it is and can be neither significantly changed nor totally avoided. There is, however, considerable room for variation in the implementation of specific curricula, notably in the examples that are used to motivate and illuminate the core. Appendix II, adapted from a report of the National Center for Research in Vocational Education (Forman and Steen 1999), offers some examples of important but neglected topics that can simultaneously reinforce mathematical concepts in the core and connect mathematics to ideas and topics in the world in which students live. Some recent textbooks (e.g., Pierce et al. 1997) build on similar ideas.

But perhaps even more important than an enriched variety of examples and topics would be a powerful emphasis on aspects of what NCTM calls process standards. As the practice of medicine involves far more than just diagnosing and prescribing, so the practice of mathematics involves far more than just deducing theorems or solving problems. It involves wide-ranging expertise that brings number and inference to bear on problems of everyday life. Part of learning mathematics is to experience the wide scope of its practice, which is what the process standards are all about.

Some aspects of mathematical practice are entirely pragmatic, dealing with real systems and situations of considerable complexity. A mathematics education should prepare students to deal with the kinds of common situations in which a mathematical perspective is most helpful. Common examples include scheduling, modeling, allocating resources, and preparing budgets. In this computer age, students also need to learn to use the tools of modern technology (e.g., spreadsheets, statistical packages, Internet resources) to collect and organize data, to represent data visually, and to convert data from one form and system to another. Performance standards for mathematics in the age of computers means performance *with* computer tools.

Other aspects of mathematics are anchored more in logic than in practice, in drawing inferences rather than working with data. Ever since Euclid, mathematics has been defined by its reliance on deductive reasoning, but there are many other kinds of reasoning in which mathematical thinking plays an important role. Students finishing high school should have enough experience with different kinds of reasoning to understand the differences between them and the appropriate role for each. Some examples include:

- *Scientific Inference:* Gathering data; detecting patterns, making conjectures; testing conjectures; drawing inferences; verifying versus falsifying theories

- *Legal Inference:* Levels of convincing argument; persuasion and counterexamples; informal inference (suspicion, experience, likelihood); legal standards (beyond reasonable doubt versus preponderance of evidence); logical trees in court decisions

- *Mathematical Inference:* Logical reasoning and deduction; assumptions and conclusions; axiomatic systems; theorems and proofs; proof by direct deduction, by indirect argument, and by "mathematical induction"; classical proofs (e.g., isosceles triangle, infinitude of primes, Pythagorean theorem)

- *Statistical Inference:* Rationale for random samples; double-blind experiments; surveys and polls; confidence intervals; causality versus correlation; multiple and hidden factors, interaction effects; judging validity of statistical claims in media reports.

Both in practical situations of planning and modeling and in the more intellectual sphere of reasoning and inference, these aspects of high school mathematics are well suited to reinforcement in other subjects. As previously noted, to build mathematical breadth and ensure lasting connections, mathematics must be taught, to some degree, in every subject and every classroom.

I have argued for parallel development of the three legs of the mathematical stool—data analysis, geometry, and algebra—to maximize interconnections that are essential for long-term learning. But in grades 9 to 11 there is yet another very practical reason: the increasing number of state-mandated tests that are often set at the tenth-grade level. These tests, if they are aligned with the goals of instruction, should treat all standards in a balanced manner. In particular, data analysis, geometry, and algebra should be equally present on tenth-grade tests; they therefore must be equally present in ninth- and tenth-grade courses. If high school courses remain layered as they are now, state examinations will continue to concentrate only on algebra and geometry, leaving data analysis out of the picture.

Options

Ideally, every student should study the same mathematics through grade 8, with only minor variation in examples to support different student interests and abilities. Accommodation to student differences in middle school should reflect student needs, not variations in anticipated career plans or college requirements. More instruction should be provided for students who need more support, more extensions for students who need greater challenges. Most students who are able to move rapidly through the core curriculum would be much better served with extensions that provide additional depth and variety than with acceleration, especially with examples that open their minds to the many connections among mathematical topics and with diverse applications. (Acceleration may be appropriate for a very few exceptionally talented students—fewer than one in a hundred—but only if they are able to pursue the entire curriculum at its deepest level. Acceleration of the core curriculum alone, without extensions, is pointless.)

In high school, student interests emerge with greater strength and legitimacy and both students and parents expect schools to provide some options. Historically, schools have tried to do this by a combination of two strategies: tracking and filtering. Both strategies amount to an abdication of educational responsibility. Weaker students were placed in commercial or general tracks that avoided algebra, thus barring them from further work in any quantitatively based field. Stronger students were immersed in a form of algebra that was designed to filter out students who did not appear capable of later success in calculus. The consequences of this strategy are well known: large numbers of students leave school both ignorant of and anxious about mathematics.

It is possible to offer options without foreclosing students' futures. Three types of very successful programs can be found in today's schools: career, academic, and scientific. The first provides rigorous preparation for the high-performance workplace, the second offers thorough preparation for college, the third offers advanced preparation for scientific careers. These programs vary in mathematical intensity and depth, but all provide students with substantial experience in data analysis, geometry, and algebra. Each leaves students prepared for work and postsecondary education but at different mathematical levels, separated by approximately one year of mathematical study.

Many educators argue vehemently against any tracks on the ground that they magnify inequities in educational advantage at a time when all students need to be equally prepared (rather than unequally prepared) for postsecondary education. Others argue that mathematically able students need a separate track to enable them to maintain their interest and fulfill their potential (Gavosto et al. 1999; Sheffield 1999). Still others argue, for similar reasons, that many students will thrive better in a career-oriented track—especially one that stresses skills required for new high-performance, technologically intensive industries (Bottoms 1993; Hoachlander 1997).

Research and practice show definitively that students learn better when they can fit new ideas into meaningful contexts (Askew and William 1995; Bransford et al. 1999). Because high school students have quite different interests, it makes sense to provide some choices in the context and setting of their mathematics courses. Moreover, given the variety of programs in higher education—ranging from technical certificates offered by community colleges to bachelor's degrees offered by liberal arts colleges, from majors in philosophy or art to hotel management and hazardous waste disposal—it is clear that student preparation can legitimately be varied.

Thus we can imagine different settings for mathematics in grades 9 to 11, each achieving the same general goals but with rather different details. If rigorously delivered, a variety of substantive programs can prepare students for postsecondary education without remediation—an important practical and political point. Whether the equations studied in algebra come from physics or automobile mechanics, or the data from economics or computer repair, what students learn about handling equations and data would be approximately the same. Differences among programs become weaknesses only if some programs leave students ill prepared for their future—either for work or for study. But options that differ primarily in specifics rather than in broad goals can serve all students well.

Providing options in grades 9 to 11 does mean that students will finish the eleventh grade at different mathematical levels—some much more advanced than others. Especially in a subject like mathematics, wide variation in achievement is inevitable. By encouraging students to relate mathematics to their personal interests, all students' learning will be enhanced, but so will variability. Success in mathematics education is not determined by the quantity of mathematics learned but by the ability and disposition to use a variety of mathematical tools in further work or study. It is far better for students to finish the required core of school mathematics willing and eager to learn more than to have mastered skills that they hope never to have to use again.

Student options become fully realized in grade 12 when mathematics itself becomes optional. Ideally, schools should offer a wide choice of further mathematics—for example, calculus-readiness, computer graphics, mathematical modeling, computer science, statistics, or biomathematics—in part to show students how pervasive mathematics really is and how many options there are for continued study. These elective courses need not be offered every year because none is uniquely necessary for students' further study and none is likely to be part of the testing associated with the school-college transition. Calculus itself can easily be left for college (except for the few students who can legitimately cover this entire program one year ahead of the others). One measure of success of a school's mathematics program would be the percentage of students who elect mathematics in twelfth grade.

Synopsis

To summarize, all middle school students (grades 6 to 8) would study a three-year, non-tracked curriculum that provides equal and tightly linked introductions to data analysis, geometry, and algebra. When students enter high school, they would move into a second three-year mathematics curriculum that may provide some options based on student interests. No matter the emphasis, however, each high school program would advance equally the three main themes (data analysis, geometry, algebra) without letting any lag behind. Different programs may emphasize different contexts, different tools, and different depths, but each would leave students prepared both for the world of work and for postsecondary education.

In this plan, minimum high school graduation requirements (certified, for example, by high-stakes state tests that are typically set at tenth-grade competence) would represent citizen-level quantitative literacy, one year behind mathematics preparation for college admission (eleventh grade), which would be one year behind qualification for mathematically intensive college programs. Aiming for this three-step outcome of high school mathematics is more logical and more achievable than the imagined (but never achieved) ideal of having every student leave high school equally educated in mathematics and equally prepared for college admission.

By studying a balanced curriculum, students would leave school better prepared for employment, more competitive with their international peers, and well positioned for a variety of postsecondary programs. By experiencing breadth and connectedness rather than depth and verticality, students would have repeated opportunities to engage mathematics afresh as their own interests and attitudes evolve. By focusing on the symbiosis of computers and mathematics, students would experience how mathematics is practiced. And by studying a blend of mathematics, statistics, and numeracy, students would be flexibly prepared for life and work in the twenty-first century.

Appendix I: Standards for School Mathematics

(Adapted from *Principles and Standards for School Mathematics,* National Council of Teachers of Mathematics 2000)

Content Standards:

Instructional programs should enable all students,

in number and operations, to

- understand numbers, ways of representing numbers, relationships among numbers, and number systems;
- understand meanings of operations and how they relate to one another;
- compute fluently and make reasonable estimates;

in algebra, to

- understand patterns, relations, and functions;
- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships;
- analyze change in various contexts;

in geometry, to

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations;
- use visualization, spatial reasoning, and geometric modeling to solve problems;

in measurement, to

- understand measurable attributes of objects and the units, systems, and processes of measurement;
- apply appropriate techniques, tools, and formulas to determine measurements;

in data analysis and probability, to

- formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them;
- select and use appropriate statistical methods to analyze data;
- develop and evaluate inferences and predictions that are based on data;
- understand and apply basic concepts of probability.

Process Standards:

Instructional programs should enable all students,

in problem solving, to

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving;

in reasoning and proof, to

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof;

in communication, to

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely;

in connections, to

- recognize and use connections among mathematical ideas;

- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics;

in representation, to

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Appendix II: Selected Topics for School Mathematics

(Adapted from *Beyond Eighth Grade: Functional Mathematics for Life and Work*, published by the National Center for Research in Vocational Education 1999)

Geometry: In addition to standard topics (e.g., measurement of figures in two- and three-dimensions, congruence and similarity, reflections and rotations, triangle trigonometry, classical proofs and constructions), students should be introduced to:

- *Dimensions:* Coordinate notation; dimension as factor in multivariable phenomena. Geometric dimensions (linear, square, and cubic) versus coordinate dimensions in multivariable phenomena. Proper versus improper analogies. Discrete versus continuous dimensions.
- *Dimensional Scaling:* Linear, square, and cubic growth of length, area, and volume; physical and biological consequences. Fractal dimensions.
- *Spatial Geometry:* Calculating angles in three dimensions (e.g., meeting of roof trusses); building three-dimensional objects and drawing two-dimensional diagrams. Interpreting construction diagrams; nominal versus true dimensions (e.g., of 2 x 4s); tolerances and perturbations in constructing three-dimensional objects.
- *Global Positioning:* Map projections, latitude and longitude, global positioning systems (GPS); local, regional, and global coordinate systems.

Data: In addition to standard topics (e.g., ratios, percentages, averages, probabilities) students should be introduced to:

- *Measurement:* Estimating weights, lengths, and areas. Direct and indirect measurement. Use of appropriate instruments (rulers, tapes, micrometers, pacing, electronic gauges, plumb lines). Squaring corners in construction. Estimating tolerances. Detecting and correcting misalignments.
- *Calculation:* Accurate paper-and-pencil methods for simple arithmetic and percentage calculations; calculator use for complex calculations; spreadsheet methods for problems with a lot of data. Use of mixed methods (mental, pencil, calculator). Strategies for checking reasonableness and accuracy. Significant digits; interval arithmetic; errors and tolerances. Accuracy of calculated measurements.
- *Mental Estimation*: Quick, routine mental estimates of costs, distances, times. Estimating orders of magnitude. Reasoning with ratios and proportions. Mental checking of calculator and computer results. Estimating unknown quantities (e.g., number of high school students in a state or number of gas stations in a city).
- *Numbers:* Whole numbers (integers), fractions (rational numbers), and irrational numbers (π, $\sqrt{2}$). Number line; mixed numbers; decimals; percentages. Prime numbers, factors; simple number theory; fundamental theorem of arithmetic. Binary numbers and simple binary arithmetic. Scientific notation; units and conversions. Number sense, including intuition about extreme numbers (lottery chances, national debt, astronomical distances).
- *Coding:* Number representations (decimal, binary, octal, and hex coding). ASCII coding; check digits. Patterns in credit card, Social Security, telephone, license plate numbers. Passwords and PINS.
- *Index Numbers:* Weighted averages. Definitions and abuses. Examples in the news: stock market averages; consumer price index; unemployment rate; SAT scores; college rankings.
- *Data Analysis:* Measures of central tendency (average, median, mode) and of spread (range, standard deviation, mid-range, quartiles, percentiles). Visual displays of data (pie charts, scatter plots, bar graphs, box and whisker charts). Quality control charts. Recognizing and dealing with outliers.
- *Probability:* Chance and randomness. Calculating odds in common situations (dice, coin tosses, card games); expected value. Random numbers; hot streaks. Binomial probability; binomial approximation of normal distribution. Computer simulations; estimating area by Monte Carlo methods. Two-way contingency tables; bias paradoxes.
- *Risk Analysis:* Estimates of common risks (e.g., accidents, diseases, causes of death, lotteries). Confounding factors. Communicating and interpreting risk.

Algebra: In addition to standard topics (e.g., variables, symbols, equations, relations, graphs, functions, slope, inequalities), students should be introduced to:

- *Algorithms:* Alternative arithmetic algorithms; flow charts; loops; constructing algorithms; maximum time versus average time comparisons.
- *Graphs:* Sketching and interpreting graphs; translating between words and graphs (and vice versa) without intervening formulas.
- *Growth and Variation:* Linear, exponential, quadratic, harmonic, and normal curve patterns. Examples of situations that fit these patterns (bacterial growth, length of day) and of those that do not (e.g., height versus weight; income distribution).
- *Financial Mathematics:* Personal finance; loans, annuities, insurance. Investment instruments (stocks, mortgages, bonds).
- *Exponential Growth:* Examples (population growth, radioactivity, compound interest) in which rate of change is proportional to size; doubling time and half-life as characteristics of exponential phenomena; ordinary and log-scaled graphs.
- *Normal Curve:* Examples (e.g., distribution of heights, repeated measurements, production tolerances) of phenomena that distribute in a bell-shaped curve and examples that do not (e.g., income, grades, typographical errors, life spans). Area as measure of probability. Meaning of 1-, 2-, and 3σ.
- *Parabolic Patterns:* Examples (falling bodies, optimization, acceleration) that generate quadratic phenomena; relation to parabolic curves.
- *Cyclic functions:* Examples (time of sunrise, sound waves, biological rhythms) that exhibit cyclic behavior. Graphs of sin and cos; consequences of $\sin^2 \theta + \cos^2 \theta = 1$.

References

Achieve Mathematics Achievement Partnership (Achieve). 2001. *Mathematics Expectations for the End of Eighth Grade.* Washington, DC: Achieve Mathematics Achievement Partnership.

American Mathematical Association of Two-Year Colleges. 1995. *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus*. Memphis, TN: American Mathematical Association of Two-Year Colleges. Retrieved 04/10/02, at www.richland.cc.il.us/imacc/standards/.

Askew, Mike, and Dylan William. 1995. *Recent Research in Mathematics Education 5–16.* London: Her Majesty's Stationery Office.

Atkinson, Richard C. 2001. "Standardized Tests and Access to American Universities." Robert H. Atwell Distinguished Lecture. Washington, DC: American Council on Education. Retrieved 04/10/02, at www.ucop.edu/ucophome/commserv/sat/speech.html.

Bailey, Thomas R. 1997. *Integrating Academic and Industry Skill Standards*. Berkeley, CA: National Center for Research in Vocational Education, University of California.

Barker, William H. 2000. *Curriculum Foundations.* Working papers for the Committee on the Undergraduate Program in Mathematics (CUPM). Washington, DC: Mathematical Association of America. Retrieved 04/10/02, at academic.bowdoin.edu/faculty/B/barker/dissemination/Curriculum_Foundations/.

Bass, Hyman. 1997. "Mathematicians as Educators." *Notices of American Mathematical Society.* 44:1 (January): 18–21. Reprinted in *Contemporary Issues in Mathematics Education*, edited by Estela A. Gavosto, Steven G. Krantz, and William McCallum, 1–6. New York, NY: Cambridge University Press, 1999.

Beaton, A. E., et al. 1996. *Mathematics Achievement in the Middle School Years: IEA's Third International Mathematics and Science Study*. Chestnut Hill, MA: Center for the Study of Testing, Evaluation, and Educational Policy, Boston College.

Bottoms, Gene. 1993. *Redesigning and Refocusing High School Vocational Studies*. Atlanta, GA: Southern Regional Education Board.

Bransford, John D., Ann L. Brown, and Rodney R. Cocking, eds. 1999. *How People Learn: Brain, Mind, Experience, and School*. National Research Council. Washington, DC: National Academy Press.

California Academic Standards Commission. 1997. *Mathematics Content Standards for California Public Schools*. Sacramento, CA: State Board of Education. Retrieved 04/10/02, at www.cde.ca.gov/standards/math/.

Campbell, J. R., et al. 1996. *NAEP 1994 Trends in Academic Progress*. Washington, DC: National Center for Education Statistics, U.S. Department of Education.

Chambers, Donald L. 1994. "The right algebra for all." *Educational Leadership,* 51:6, pp. 85–86.

Cheney, Lynne. 1997. "Creative Math or Just 'Fuzzy Math'? Once Again, Basic Skills Fall Prey to a Fad." *New York Times,* 11 August 1997: A13.

Compaine, Benjamin M., ed. 2001. *The Digital Divide*. Cambridge, MA: Massachusetts Institute of Technology Press.

Crosswhite, F. Joe, et al. 1986. *Second International Mathematics Study: Report for the United States* (2 vols.). International Association for the Evaluation of Educational Achievement. Champaign, IL: Stipes.

Devlin, Keith. 1994. *Mathematics: The Science of Patterns.* New York, NY: Scientific American Library.

Dossey, John, and Zalman Usiskin. 2000. *Mathematics Education in the United States—2000: A Capsule Summary*. Washington, DC: United States National Commission on Mathematics Instruction.

Education Trust. 1999. *You Can't Get There from Here: A Study of Tests from High School through Postsecondary Admission and Placement*. Washington DC: Education Trust.

Flanders, James. 1987. "How Much of the Content in Mathematics Textbooks Is New?" *Arithmetic Teacher* 35(1): 18-23.

Forman, Susan L., and Lynn Arthur Steen. 1998. *Beyond Eighth Grade: Report of a Workshop on Mathematical and Occupational Skill Standards*. Berkeley, CA: National Center for Research in Vocational Education.

Forman, Susan L., and Lynn Arthur Steen. 1999. *Beyond Eighth Grade: Functional Mathematics for Life and Work*. Berkeley, CA: National Center for Research in Vocational Education. Reprinted in *Learning Mathematics for a New Century,* edited by Maurice Burke, 127–57. Reston, VA: National Council of Teachers of Mathematics, 2000. Retrieved 04/10/02, at ncrve.berkeley.edu/abstracts/MDS-1241/.

Gardner, Howard. 1999. "With Pencils Sharpened: Taking Tests to Task in America." *Harvard Magazine* (November–December): 29–35.

Gavosto, Estela A., Steven G. Krantz, and William McCallum. 1999. *Contemporary Issues in Mathematics Education*. MSRI Publications, vol. 36. New York, NY: Cambridge University Press.

Groves, Martha. "Exit-Exam Retreats Reflect Fears That Students Lack Skills." *Los Angeles Times* 17 December 2000.

Hammond, Allen. 1978. "Mathematics—Our Invisible Culture." In *Mathematics Today: Twelve Informal Essays,* edited by Lynn Arthur Steen, 15–34. New York, NY: Springer-Verlag.

Hembree, Ray, and Donald J. Dessart. 1992. "Research on Calculators in Mathematics Education." In *Calculators in Mathematics Education,* edited by James T. Fey, 23–32. Reston, VA: National Council of Teachers of Mathematics.

Hirsch, Christian R., and Marilyn J. Zweng, eds. 1985. *The Secondary School Mathematics Curriculum*. Reston, VA: National Council of Teachers of Mathematics.

Hoachlander, Gary. 1997. "Organizing Mathematics Education Around Work." In *Why Numbers Count: Quantitative Literacy for Tomorrow's America,* edited by Lynn Arthur Steen, 122–36. New York, NY: College Entrance Examination Board.

Hoaglin, David C., and Davis S. Moore, eds. 1992. *Perspectives on Contemporary Statistics*. Washington, DC: Mathematical Association of America.

Hoffrage, Ulrich, Samuel Lindsey, Ralph Hartwig, and Gerd Gigerenzer. 2000. "Communicating Statistical Information." *Science* 290 (11 December): 2261–62.

Howson, A. Geoffrey and Bryan Wilson, eds. 1986. *School Mathematics in the 1990s*. ICMI Study Series. Cambridge, UK: Cambridge University Press.

Howson, A. Geoffrey, Jean-Pierre Kahane, P. Lauginie, E. de Turckheim., eds. 1988. *Mathematics as a Service Subject*. ICMI Study Series. Cambridge, UK: Cambridge University Press.

Howson, Geoffrey. 1991. *National Curricula in Mathematics*. Leister, UK: The Mathematical Association.

International Association for the Evaluation of Educational Achievement (IEA). 2000. *TIMSS 1999: Trends in Mathematics and Science Achievement Around the World*. Boston, MA: Boston College International Study Center. Retrieved 04/10/02, at timss.bc.edu/timss1999.html.

International Association for the Evaluation of Educational Achievement (IEA). 2001. *TIMSS 1999 Benchmarking Report*. Boston, MA: Boston College International Study Center. www.timss.org/timss1999b/mathbench_report/t99b_math_report.html.

International Technology Education Association (ITEA). 2000. *Standards for Technological Literacy: Content for the Study of Technology*. Reston, VA: International Technology Education Association.

Jackson, Allyn. 1997. "The Math Wars," Parts I and II. *Notices of American Mathematical Society.* 44 (June/July, August): 619–702; 817–27.

Kenney, P. A., and Edward A. Silver, eds. 1996. *Results from the Sixth Mathematics Assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.

Kilpatrick, Jeremy. 1997. "Confronting Reform." *The American Mathematical Monthly* 104(10) (December): 955–62.

Lacampagne, Carole B. William Blair, and Jim Kaput., eds. 1995. *The Algebra Initiative Colloquium* (2 vols.). Washington, DC: Office of Educational Research and Improvement, U.S. Department of Education.

Lapointe, A. E., N. A. Mead, and G. W. Phillips. 1989. *A World of Differences: An International Assessment of Mathematics and Science*. Princeton, NJ: Educational Testing Service.

Malcom, Shirley. 1997. "Making Mathematics the Great Equalizer." In *Why Numbers Count: Quantitative Literacy for Tomorrow's America,* edited by Lynn Arthur Steen, 30–35. New York, NY: College Entrance Examination Board.

Mathematical Sciences Education Board (MSEB). 1990. *Reshaping School Mathematics: A Philosophy and Framework for Curriculum.* Washington, DC: National Research Council.

Mathematical Sciences Education Board (MSEB). 1995. *Mathematical Preparation of the Technical Work Force.* Washington, DC: National Research Council.

Mathematical Sciences Education Board (MSEB). 1998. *High School Mathematics at Work: Essays and Examples for the Education of all Students.* Washington, DC: National Academy Press.

McKnight, Curtis C., F. Joe Crosswhite, John A Dossey, Edward Kifer, Jane O. Swafford, Kenneth J. Travers, and Thomas J. Cooney. 1987. *The Underachieving Curriculum: Assessing U.S. School Mathematics from an International Perspective*. Champaign, IL: Stipes.

Moore, David S., and George W. Cobb. 2000. "Statistics and Mathematics: Tension and Cooperation." *The American Mathematical Monthly* 107(7) 615–30.

Moore, Eliakim Hastings. 1903. "On the Foundations of Mathematics." *Science* 217: 401–16.

Moses, Robert. 1995. "Algebra, The New Civil Right." In *The Algebra Initiative Colloquium,* vol. II edited by Carol Lacampagne, et al., 53–67. Washington, DC: Office of Educational Research and Improvement, U.S. Department of Education.

Murnane, Richard, and Frank Levy. 1996. *Teaching the New Basic Skills: Principles for Educating Children to Thrive in a Changing Economy*. New York, NY: Free Press.

National Center for Education Statistics. 1998. *Pursuing Excellence: A Study of U.S. Twelfth-Grade Mathematics and Science Achievement in International Context*. Washington, DC: Office of Educational Research and Improvement, U.S. Department of Education. Retrieved 04/10/02, at nces.ed.gov/pubsearch/pubsinfo.asp?pubid=98049.

National Center for Education Statistics. 2000. *NAEP 1999 Trends in Academic Progress: Three Decades of Student Performance*. Washington, DC: Office of Educational Research and Improvement, U.S. Department of Education. Retrieved 04/10/02, at nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2000469.

National Commission on Excellence in Education. 1983. *A Nation at Risk: The Imperative for Educational Reform*. Washington, DC: U.S. Government Printing Office. Retrieved 04/10/02, at www.ed.gov/pubs/NatAtRisk/.

National Commission on Mathematics and Science Teaching for the 21st Century. 2000. *Before It's too Late*. Washington, DC: U.S. Department of Education.

National Council of Teachers of Mathematics. 1989. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics. Retrieved 04/10/02, at standards.nctm.org/Previous/CurrEvStds/index.htm.

National Council of Teachers of Mathematics. 2000. *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics. Retrieved 04/10/02, at standards.nctm.org/document/index.htm.

National Council of Teachers of Mathematics and Mathematical Sciences Education Board (MSEB). 1998. *The Nature and Role of Algebra in the K–14 Curriculum.* Washington, DC: National Research Council.

National Research Council (NRC). 1999. *Being Fluent with Information Technology*. Washington, DC: National Academy Press.

National Science Board. 1996. *Science and Engineering Indicators 1996*. Washington, DC: National Science Foundation.

National Skills Standards Board (NSSB). 1998. *Skills Standards and Initiatives*. Washington, DC: National Skills Standards Board. Retrieved 04/10/02, at www.nssb.org.

Nisbett, Richard E., et al. 1987. "Teaching Reasoning." *Science* 238 (30 October): 625–31.

Nohda, Nobuhiko, Shizumi Shimizu, Kohzoh Tsubota, Akira Suzuki, and Kazuhiko Murooka. 2000. *School Mathematics in Japan*. Makuhari, Japan: Japan Society of Mathematical Education.

Norris, Pippa. 2001. *Digital Divide: Civic Engagement, Information Poverty, and the Internet Worldwide*. New York, NY: Cambridge University Press.

Oakes, Jeannie. 1985. *Keeping Track: How Schools Structure Inequality*. New Haven, CT: Yale University Press.

Oakes, Jeannie. 1990. *Multiplying Inequalities: The Effects of Race, Social Class, and Tracking on Opportunities to Learn Mathematics and Science*. Santa Monica, CA: Rand Corporation.

Odom, William E. 1998. *Report of the Senior Assessment Panel for the International Assessment of the U.S. Mathematical Sciences*. Washington, DC: National Science Foundation. Retrieved 04/10/02, at www.nsf.gov/cgi-bin/getpub?nsf9895.

Packer, Arnold. 1997. "Mathematical Competencies that Employers Expect." In *Why Numbers Count: Quantitative Literacy for Tomorrow's America,* edited by Lynn Arthur Steen, 137–54. New York, NY: College Entrance Examination Board.

Paulos, John Allen. 1988. *Innumeracy: Mathematical Illiteracy and Its Consequences.* New York, NY: Hill and Wang.

Pearlman, Robert. 2001. "Inside the Digital Divide: Connecting Youth to Opportunities in the New Economy." Digital Divide Network. Retrieved 04/10/02, at www.digitaldividenetwork.org/content/stories/index.cfm?key=224.

Pierce, Don, Ed Wright, and Leon Roland. 1997. *Mathematics for Life: A Foundation Course for Quantitative Literacy*. Upper Saddle River, NJ: Prentice Hall.

Raimi, Ralph A., and Lawrence S. Braden. 1998. *State Mathematics Standards*. Washington, DC: Thomas B. Fordham Foundation. Retrieved 04/10/02, at www.edexcellence.net/standards/math.html.

Sacks, Peter. 2000. *Standardized Minds: The High Price of America's Testing Culture and What We Can Do to Change It*. Cambridge, MA: Perseus.

Schmidt, William H., Curtis C. McKnight, and Senta A. Raizen. 1996. *A Splintered Vision: An Investigation of U.S. Science and Mathematics Education*. Dordrecht: Kluwer.

Schmidt, William H., et al. 1997. *Many Visions, Many Aims, Volume 1. A Cross National Investigation of Curricular Intentions in School Mathematics*. Dordrecht: Kluwer.

Secada, Walter G. 1992. "Race, Ethnicity, Social Class, Language, and Achievement in Mathematics." In *Handbook of Research on Mathematics Teaching and Learning,* edited by D. A. Grouws, 623–60. New York, NY: Macmillan.

Secretary's Commission on Achieving Necessary Skills (SCANS). 1991. *What Work Requires of Schools: A SCANS Report for America 2000*. Washington, DC: U.S. Department of Labor.

Sheffield, Linda Jensen. 1999. *Developing Mathematically Promising Students*. Reston, VA: National Council of Teachers of Mathematics.

Shrag, Peter. 2000. "High Stakes Are for Tomatoes." *Atlantic Monthly* 286(2) (August): 19–21. Retrieved 04/10/02, at www.theatlantic.com/issues/2000/08/schrag.htm.

Silver, Edward A. 1997. "Algebra for All: Increasing Student Access to Algebraic Ideas, Not Just Algebra Courses." *Mathematics Teaching in the Middle School* 2(4): 204–207.

Silver, Edward A. 1998. *Improving Mathematics in Middle School: Lessons from TIMSS and Related Research*. Washington, DC: Office of Educational Research and Improvement, U.S. Department of Education.

Steen, Lynn Arthur. 1988. "The Science of Patterns." *Science* 240 (29 April): 611–16.

Steen, Lynn Arthur. 1990. "Numeracy." *Daedalus* 119(2) (Spring): 211–31.

Steen, Lynn Arthur. 1992. "Does Everybody Need to Study Algebra?" *Mathematics Teacher* 85 (4) (April): 258–60. Also printed in *Basic Education* 37 (January 1992): 9–13.

Steen, Lynn Arthur, and Susan L. Forman. 1995. "Mathematics for Work and Life." In *Seventy-Five Years of Progress: Prospects for School Mathematics,* edited by Iris Carl, 219–41. Reston, VA: National Council of Teachers of Mathematics.

Steen, Lynn Arthur, ed. 1997. *Why Numbers Count: Quantitative Literacy for Tomorrow's America*. New York, NY: College Entrance Examination Board.

Steen, Lynn Arthur. 1998. "Numeracy: The New Literacy for a Data-Drenched Society." *Educational Leadership* 47(2) (October): 8–13.

Steen, Lynn Arthur. 1999. "What's the Rush: Algebra for All in Eighth Grade?" *Middle Matters* 8(1) (Fall): 1–6, 7.

Steen, Lynn Arthur. "Reading, Writing, and Numeracy." 2000. *Liberal Education* 86(2) (Spring): 26–37.

Steen, Lynn Arthur, ed. 2001. *Mathematics and Democracy: The Case for Quantitative Literacy*. Princeton, NJ: National Council on Education and the Disciplines. Retrieved 04/10/02, at www.woodrow.org/nced/mathematics___democracy.html.

Stevenson, Harold W. 1998. *A TIMSS Primer: Lessons and Implications for U.S. Education*. Washington, DC: Thomas R. Fordham Foundation. Retrieved 04/10/02, at www.edexcellence.net/library/timss.html.

Stotsky, Sandra, ed. 2000. *What's at Stake in the K–12 Standards Wars: A Primer for Educational Policy Makers*. New York, NY: Peter Lang.

Usiskin, Zalman. 1995. "Why Is Algebra Important to Learn?" *American Educator* (Spring): 30–37.

Waits, Bert, and Franklin Demana. 2000. "Calculators in Mathematics Teaching and Learning: Past, Present, and Future." In *Learning Mathematics for a New Century,* edited by Maurice Burke, 51–66. Reston, VA: National Council of Teachers of Mathematics.

Wallace, Dorothy. 1999. "If I Could Talk to the Animals." In *Contemporary Issues in Mathematics Education,* edited by Estela A. Gavosto, Steven G. Krantz, and William McCallum, 39–43. New York, NY: Cambridge University Press.

Wu, Hung-Hsi. 1996. "The Mathematician and the Mathematics Education Reform." *Notices of American Mathematical Society* 43(12) (December): 1531–37.

Wu, Hung-Hsi. 1997. "The Mathematics Education Reform: Why You Should Be Concerned and What You Can Do." *The American Mathematical Monthly* 104(10) (December): 946–54.