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# Exploring Lognormal Incomes

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**Augsburg College**
**Editor: www.StatLit.org**
**US Rep: International Statistical Literacy Project**

**11 October 2014**
**National Numeracy Network**

**www.StatLit.org/pdf/2014-Schield-NNN2-Slides.pdf**
**www.StatLit.org/Excel/Create-LogNormal-Incomes-Excel2013.xlsx**

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## Log-Normal Distributions

A Log-Normal distribution is generated from a normal with mu = Ln(Median) and sigma = Sqrt[2*Ln(Mean/Median)].

The lognormal is always positive and right-skewed.

Examples:
- Incomes (bottom 97%), assets, size of cities
- Weight and blood pressure of humans (by gender)

Benefit:
- calculate the share of total income held by the top X%
- calculate Gini Coefficient,
- explore effects of change in mean-median ratio.

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## Log-Normal Distributions

"In many ways, it [the Log-Normal] has remained the Cinderella of distributions, the interest of writers in the learned journals being curiously sporadic and that of the authors of statistical test-books but faintly aroused."

"We … state our belief that the lognormal is as fundamental a distribution in statistics as is the normal, despite the stigma of the derivative nature of its name."

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## Lognormal and Excel

Use Excel to focus on the model and the results.

Excel has two Log-Normal functions:

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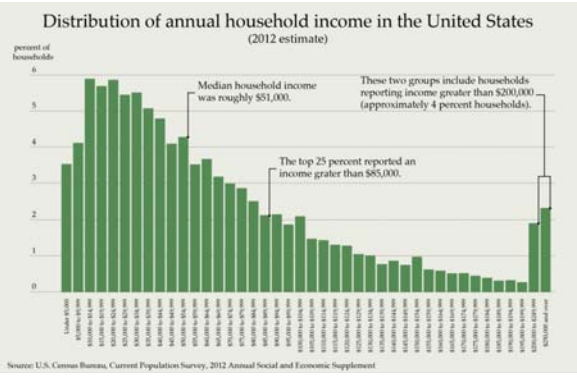


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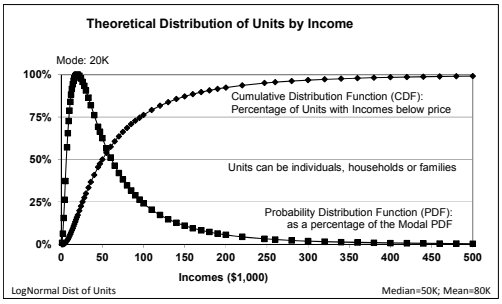

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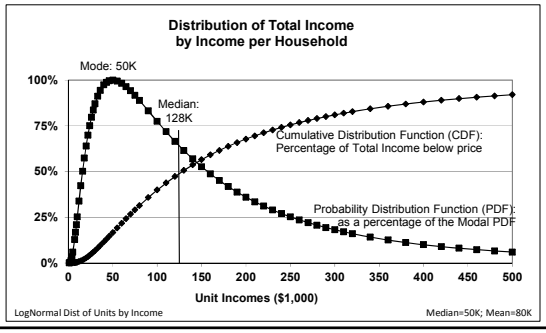


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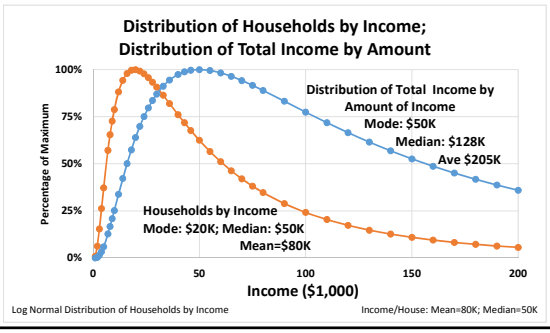


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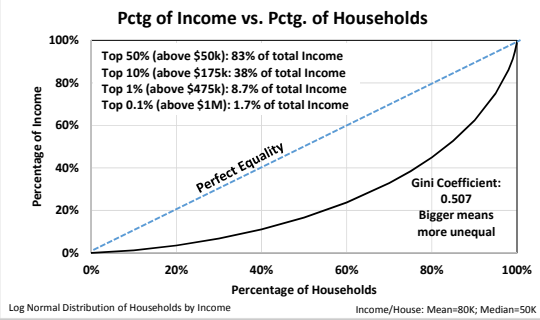


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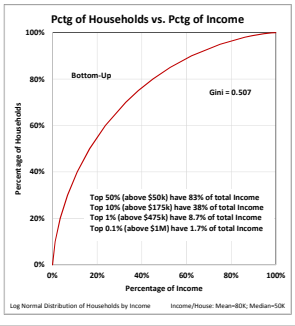

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Log-normal distribution. Median HH income: $50K.

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| Mean# | Min$ | %Income | Min$ | %Income | Gini |
| 55 | 103 | 11% | 138 | 2.9% | 0.24 |
| 60 | 135 | 15% | 204 | 4.2% | 0.33 |
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| 90 | 298 | 29% | 623 | 10.7% | 0.56 |

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| Median fixed at $50K | | | | Top 5% Households | | |
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Does this mean the poor get poorer as the rich get richer when median Income stays constant?

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## Share of Top 10%, Bottom 40% and their Palma Ratio

Palma ratio: [Share of top10%] / [Share of bottom 40%]. Cobham and Sumner (2014) argue that the Palma ratio is a more understandable measure of inequality than the Gini.

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| 85 | 187 | 40% | 39 | 10% | 4.0 | 0.53 |
| | | Median Income: $50K | | | | |

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Palma and Gini are independent of the Median Income when the Mean-Median Income ratio is constant.

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| 70 | 1.5 | 105 | 222 | 35% | 56 | 12% | 2.83 | 0.48 |
| 80 | 1.5 | 120 | 254 | 35% | 64 | 12% | 2.83 | 0.48 |
| 90 | 1.5 | 135 | 285 | 35% | 72 | 12% | 2.83 | 0.48 |
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| Constant Mean-Median Ratio | | | | | | | | |

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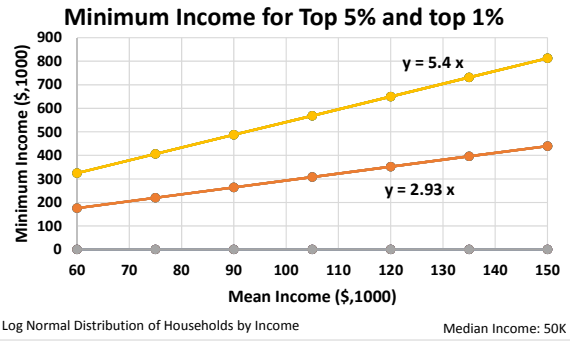


Log Normal Distribution of Households by Income

Median Income: 50K

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## Which parameters best model US household incomes?

US Median Income (Table 691*)
- $46,089 in 1970; $50,303 in 2008

Share of Total Income by Top 5% (Table 693*)
- 16.6% in 1970; 21.5% in 2008

Best log-normal fits:
- 1970 Median 46K, Mean 53K: Ratio = 1.15
- 2008 *Median 50K, Mean 73K*; Ratio = 1.46

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## Distinguish whole & part

Consider a lognormal distribution of family incomes with a median of $50K and a mean of $80K.

What percentage
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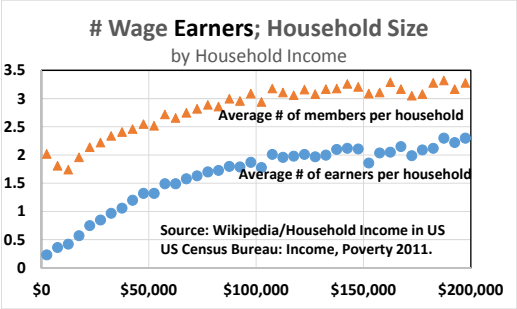


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## Conclusion

Using the LogNormal distributions provides a principled way students can explore a plausible distribution of incomes.

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## Bibliography

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Wikipedia. LogNormal Distribution.

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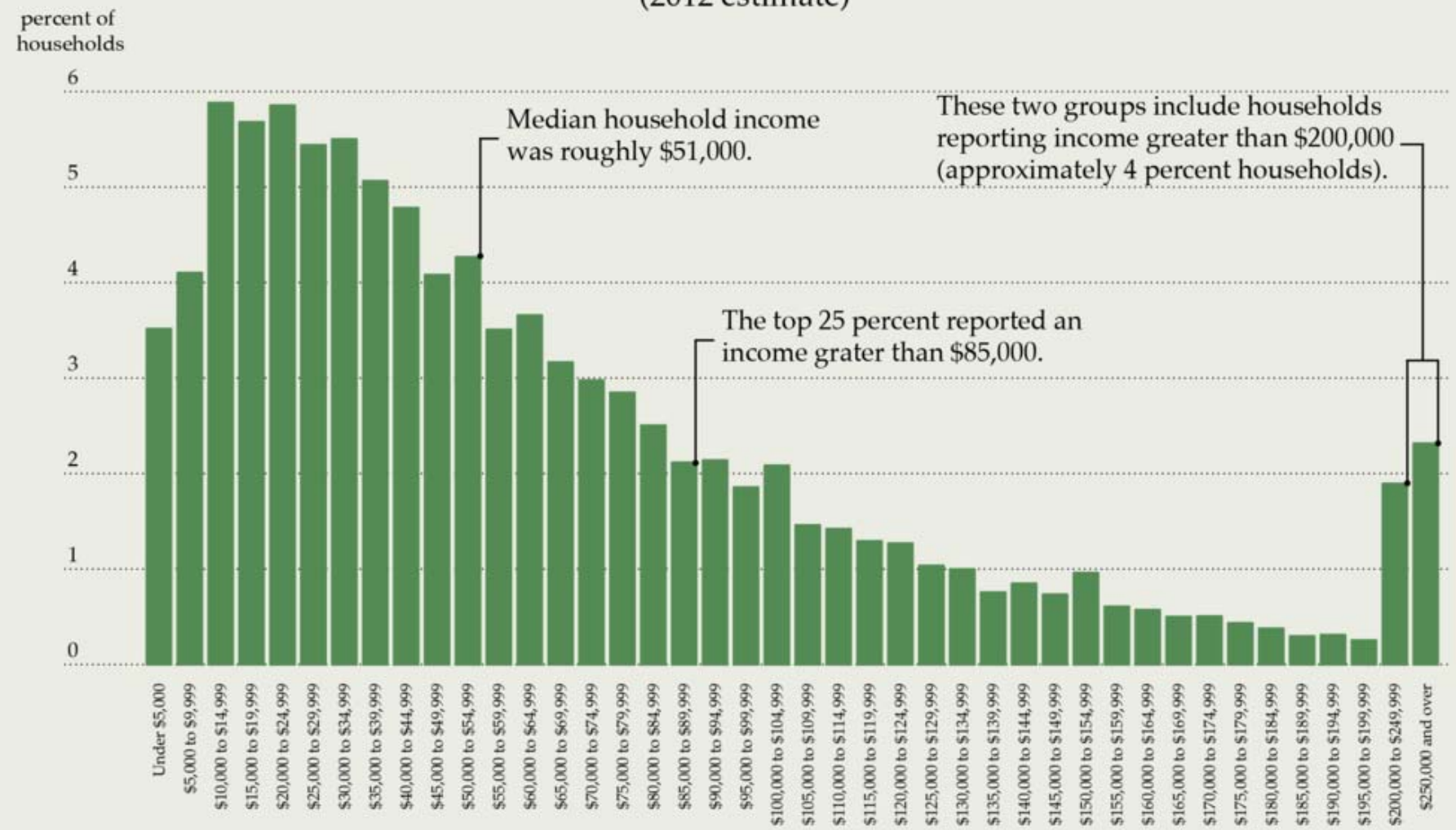
# Distribution of annual household income in the United States

(2012 estimate)




Source: U.S. Census Bureau, Current Population Survey, 2012 Annual Social and Economic Supplement

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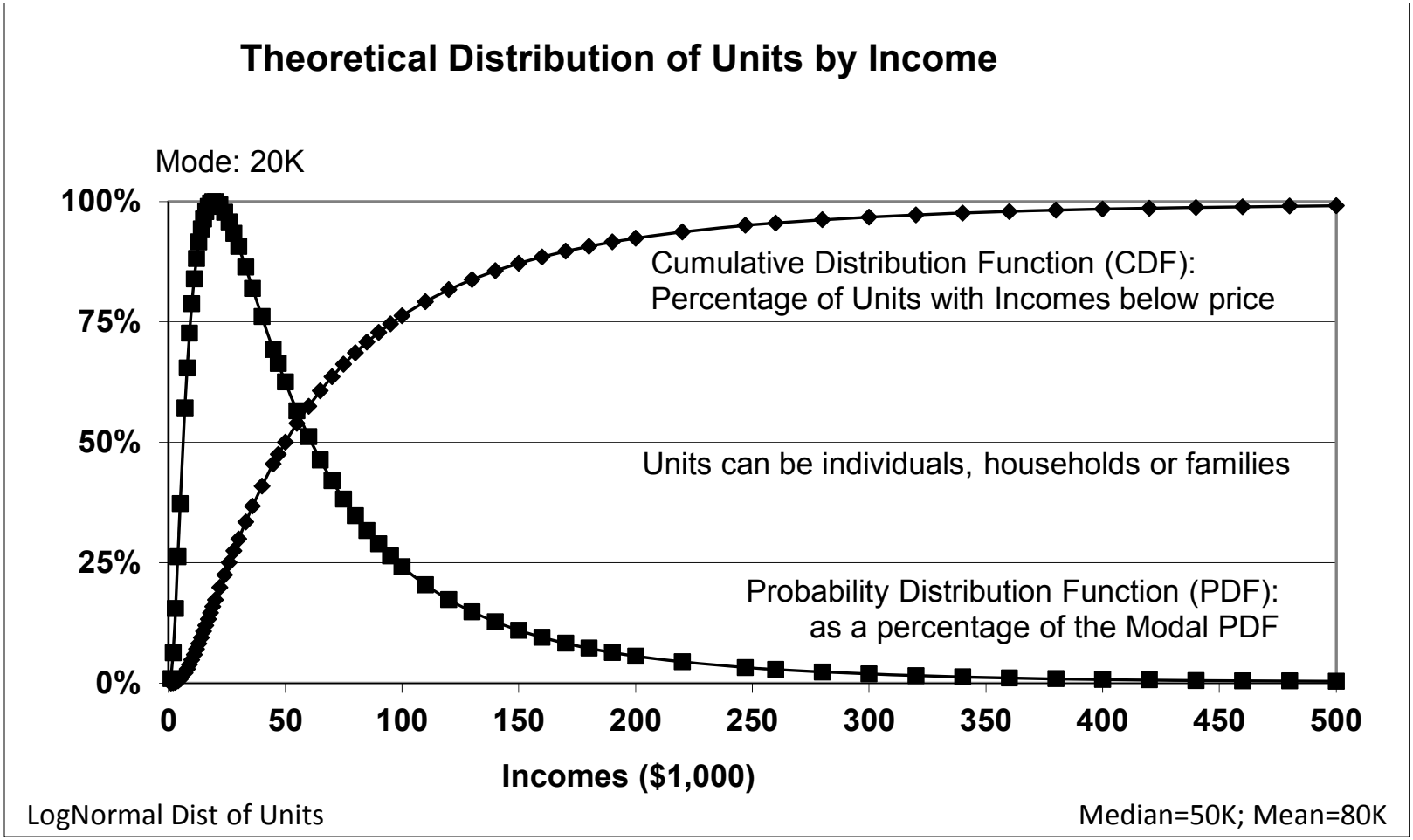

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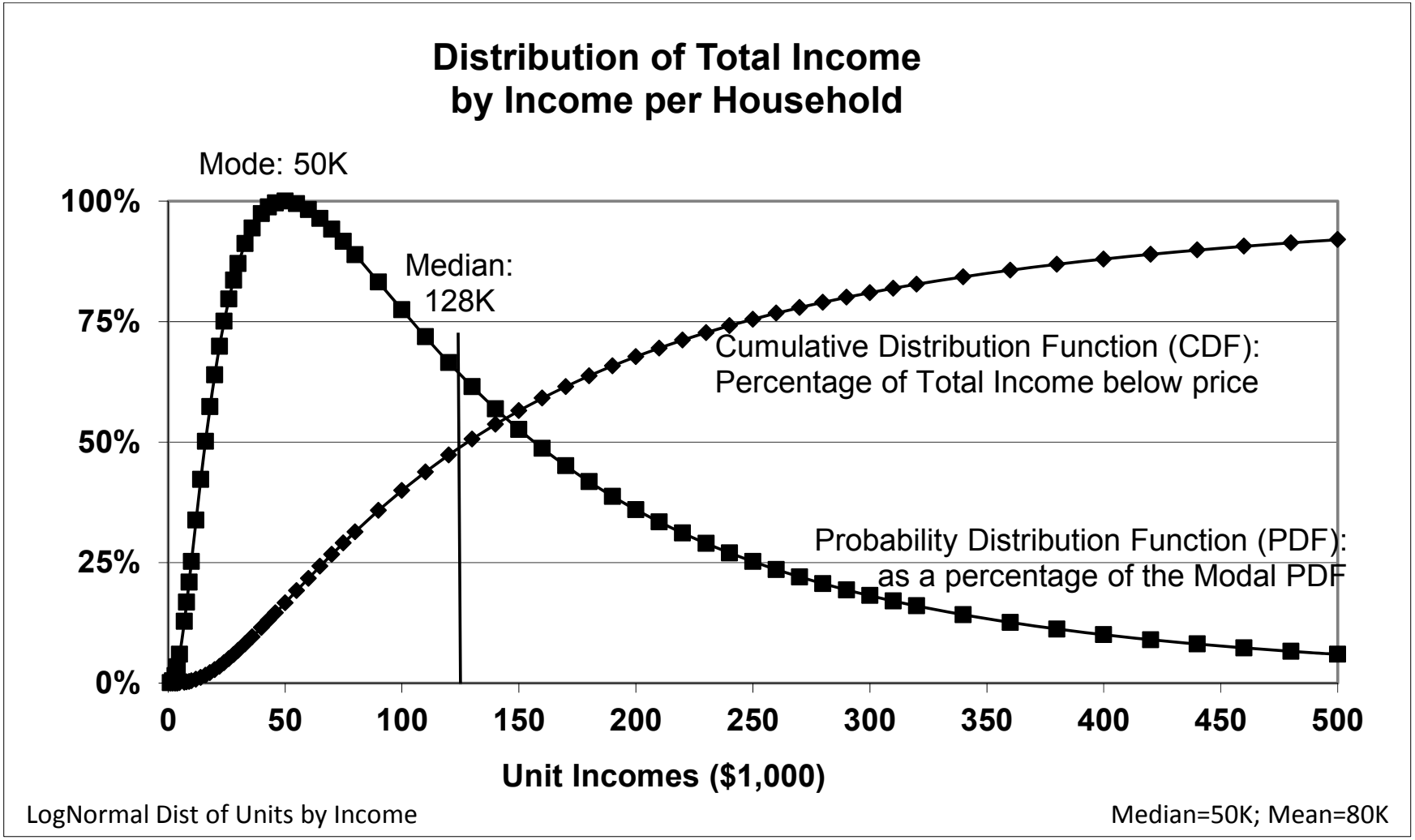


LogNormal Dist of Units by Income

Median=50K; Mean=80K

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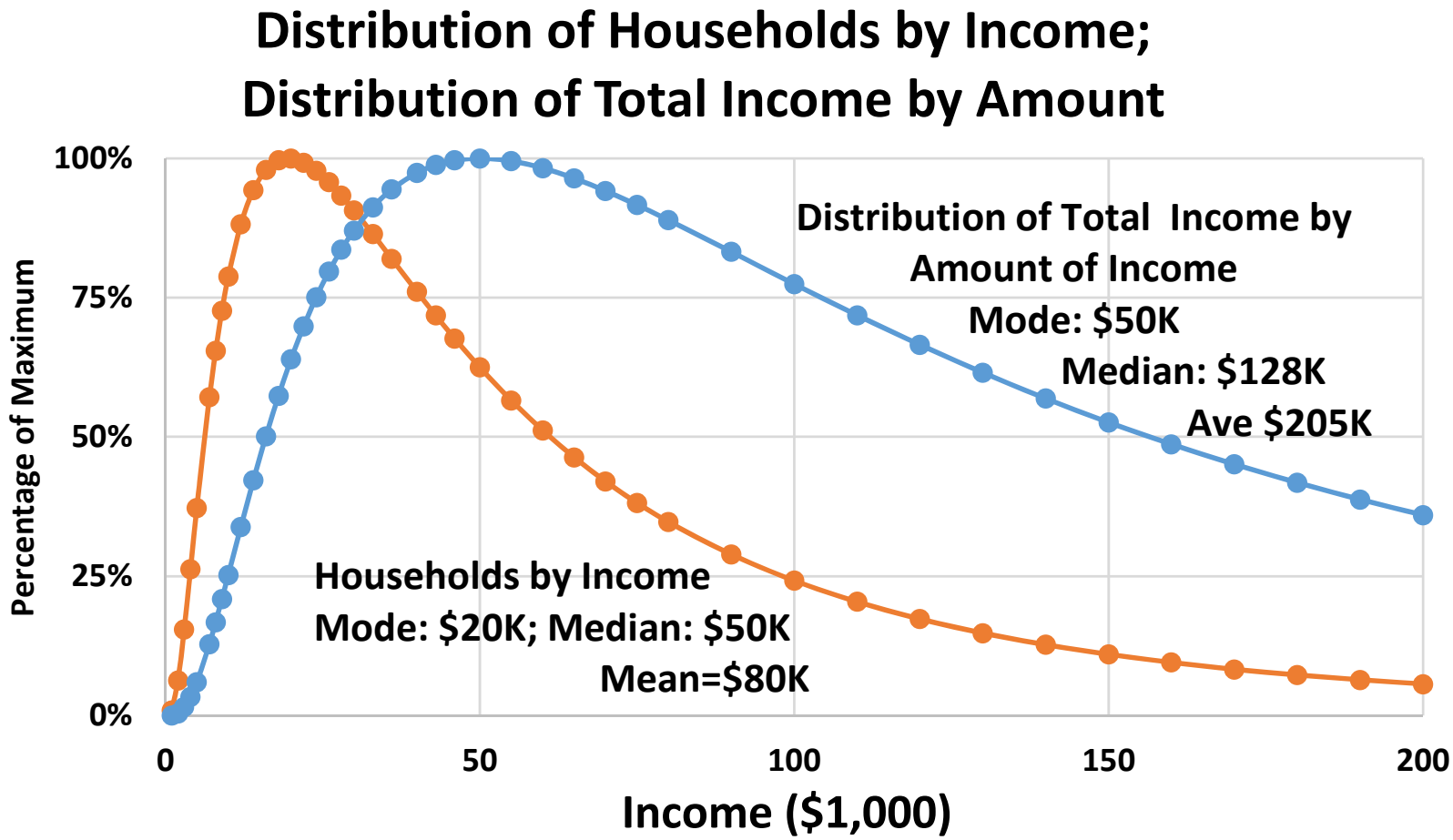


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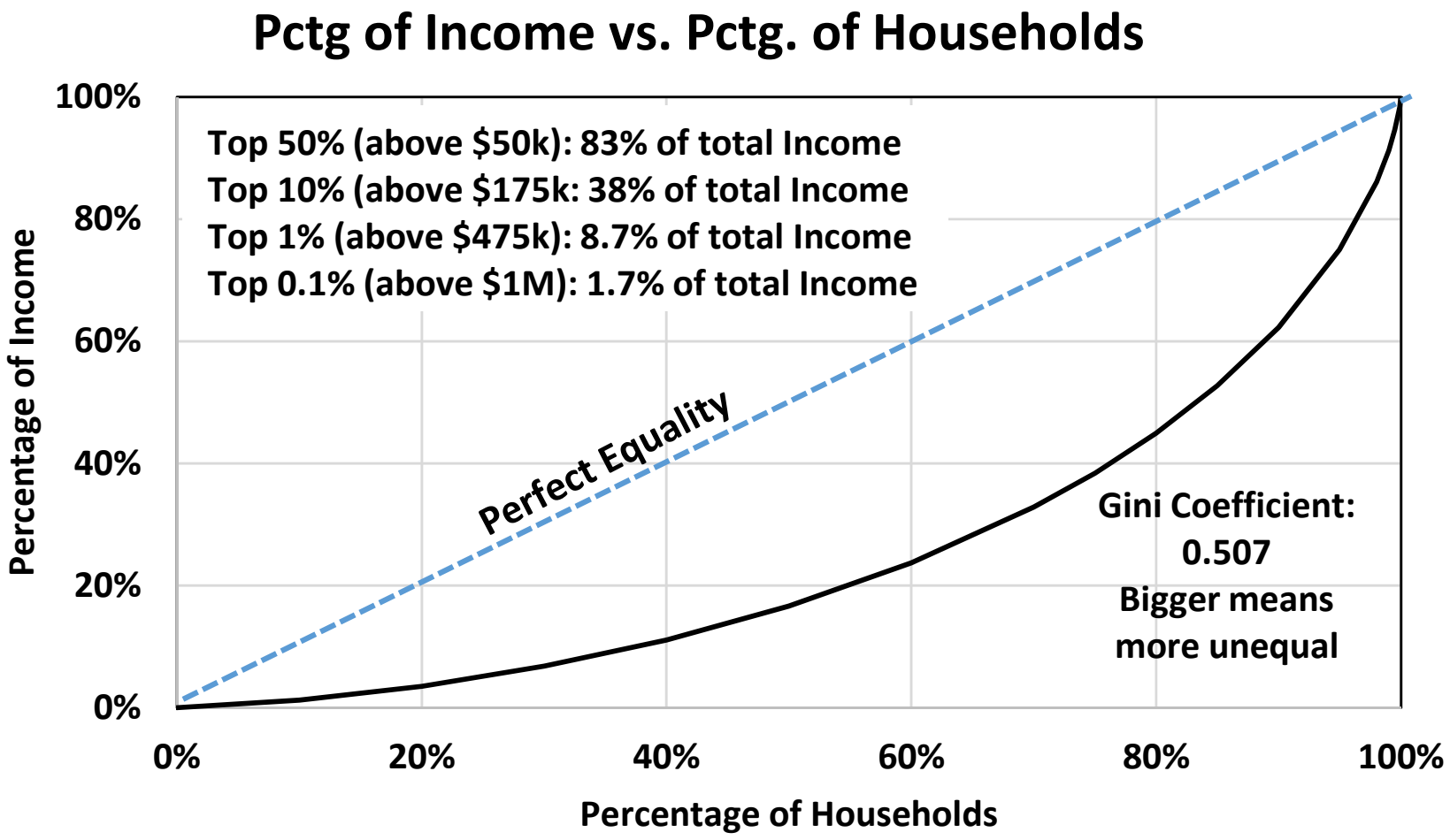


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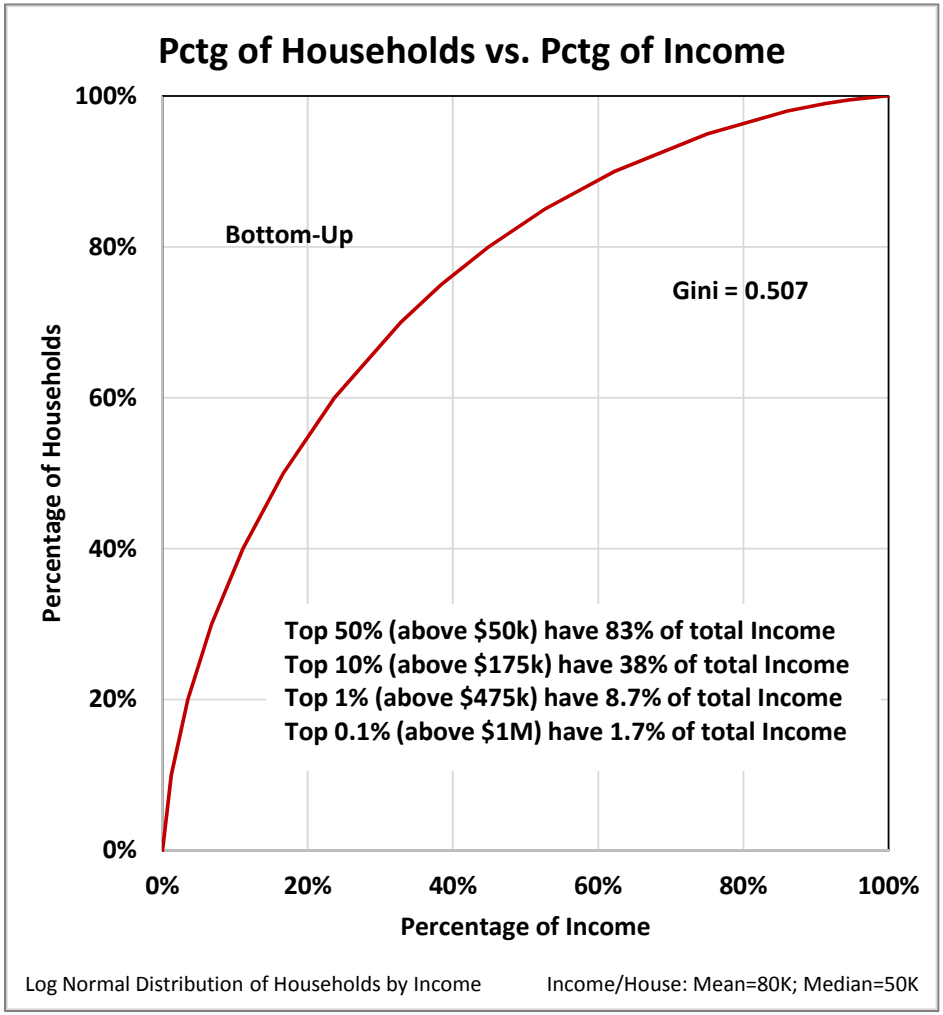

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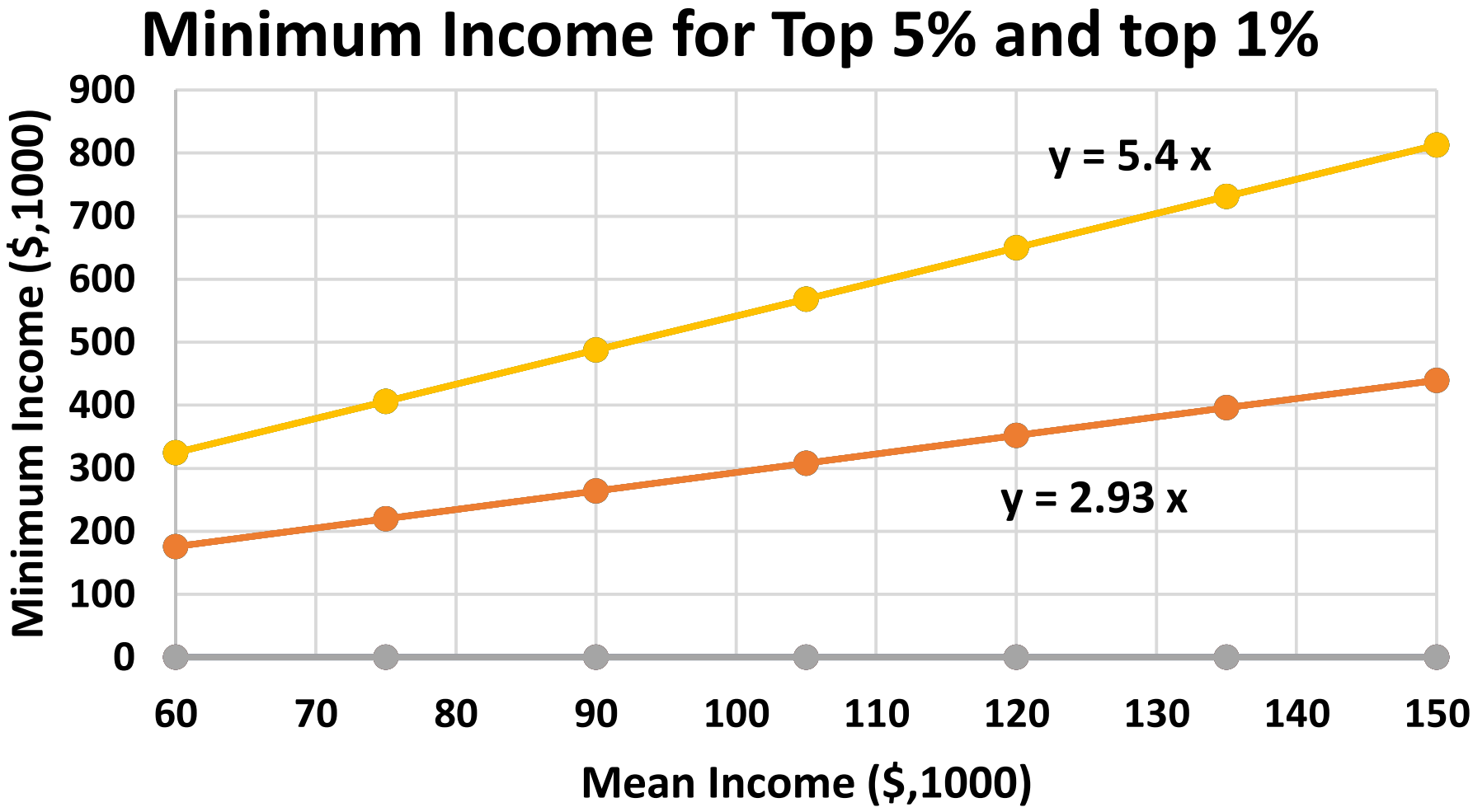


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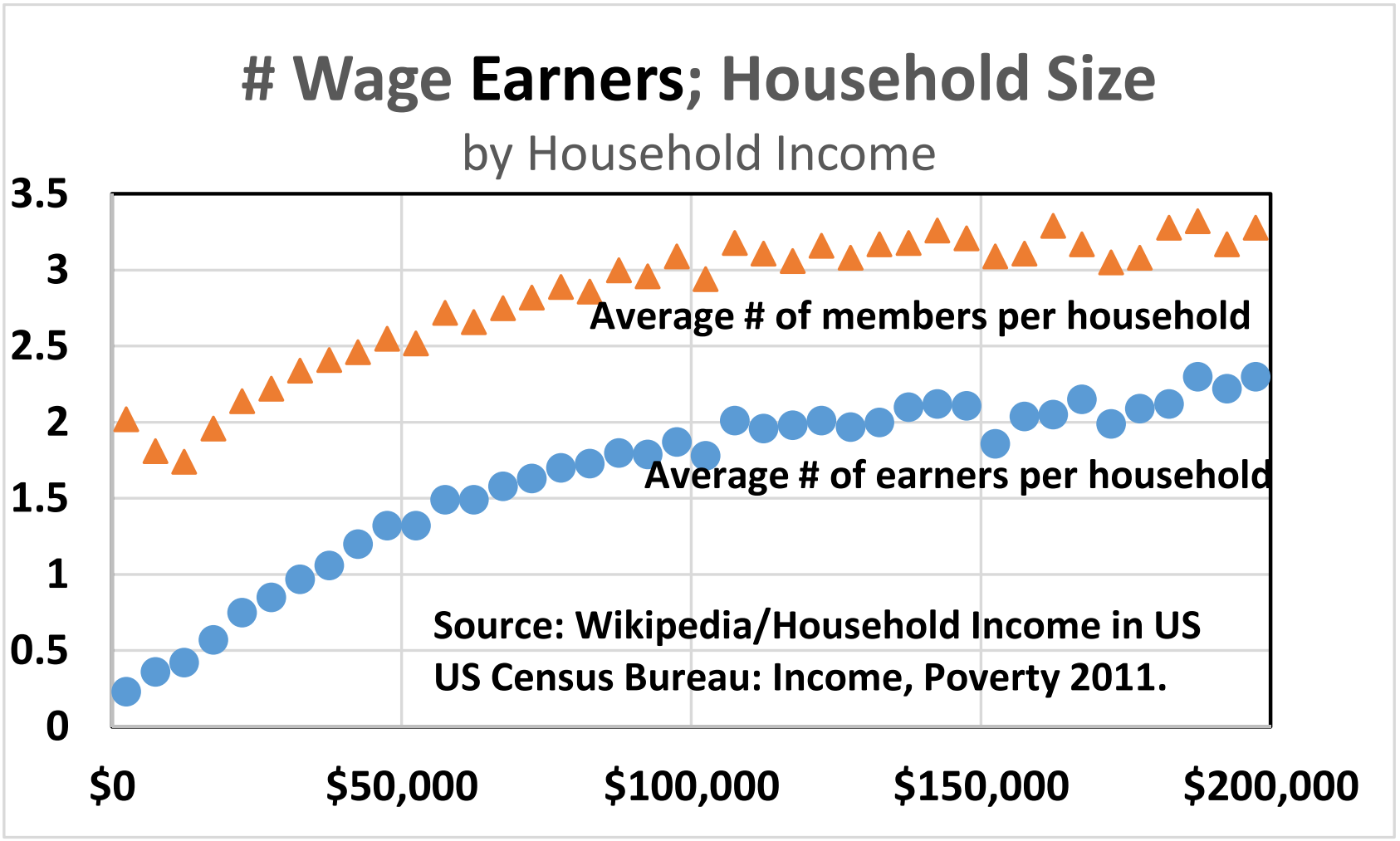

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